EMBEDDING POINTED CURVES IN K3 SURFACES

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1. Introduction

Let D be a smooth projective curve of genus g over an algebraically closed field of characteristic zero. Let $\mathcal{M}_g$ denote the moduli stack of such curves. Let (S,h) be a polarized K3 surface of genus g, i.e., h is ample and primitive with $h^2 = 2g-2$. Let $\mathcal{F}_g$ denote the moduli stack of such surfaces. Now suppose $D \subset S$ with $[D] = h$; let $\mathcal{P}_g$ denote the moduli space of such pairs (S,D), $\varphi_g : \mathcal{P}_g \to \mathcal{M}_g$ the forgetful map, and $\mathcal{K}_g \subset \mathcal{M}_g$ its image. These morphisms have been studied systematically by Mukai; we review this in more detail in Section 2. One of the highlights of this theory is the *birationality* of φ_g when $g = 11$.

We propose variations on this construction. Let $\mathcal{F}_\Lambda$ denote a moduli space of lattice polarized K3 surfaces, where $\Lambda \supset \langle h, R\rangle$ with h a polarization of degree $2g-2$ and R an indecomposable (-2)-class (smooth rational curve) with $n := h \cdot R > 0$. Let $\mathcal{P}_\Lambda$ denote the space of pairs (S,D), where $S \in \mathcal{F}_\Lambda$ and $D \in |h|$ is smooth and meets R transversally. Thus we have

$$\begin{array}{rcl} \varphi_\Lambda : \mathcal{P}_\Lambda & \to & \mathcal{M}_{g;n} := \mathcal{M}_{g,n}/\mathfrak{S}_n \\ (S,D) & \mapsto & (D; D \cap R) \end{array}$$

as well as the morphism keeping track of the pointed rational curve

$$\begin{array}{rcl} \widehat{\varphi_\Lambda} : \mathcal{P}_\Lambda & \to & (\mathcal{M}_{g,n} \times \mathcal{M}_{0,n})/\mathfrak{S}_n \\ (S,D) & \mapsto & ((D; D \cap R), (R; D \cap R)) \end{array}$$

Suppose that $\Lambda = \langle h, R\rangle$ and the source and target moduli spaces have the same dimension; thus $2g + n = 21$ in the first case and $g + n = 12$ in the second. What are $\deg(\varphi_\Lambda)$ and $\deg(\widehat{\varphi_\Lambda})$? We answer these questions for $g = 7$; see Remark 8 and Section 8.

Work of Green and Lazarsfeld [GL87] shows how lattice polarizations control the Brill-Noether properties of curves on K3 surfaces. For example, a smooth curve $D \subset S$ admits a g^1_d with $2d \le g+1$ only if there is an elliptic fibration $S \to \mathbb{P}^1$ with D as a multisection of degree d. Hence for suitable lattice polarizations φ_Λ maps to Brill-Noether strata of $\mathcal{M}_g$. Our variation is relevant to understanding the specialization of Mukai's theory over these strata.

In this note, we focus on specific lattices with $g = 7$ and 11; see Theorems 9 and 5. The approach is to specialize Mukai's construction in genus 11 to the pentagonal locus, degenerate this to a carefully chosen stable curve of genus 11, and then deform a related stable curve of genus 7 together with the ambient K3 surface.

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Our motivation for considering this geometry comes from arithmetic. The constructions in this paper have strong implications for the structure of spaces of sections for del Pezzo surface fibrations over $\mathbb{P}^1$. Details appear in [HT12].

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2. Geometric background

Consider the moduli space $M_{2,h,s}(S)$ of rank-two simple sheaves E on a K3 surface S with $c_1(E) = h$ and $\chi(E) = 2 + s$. This is holomorphic symplectic of dimension $h^2 - 4s + 2 = 2g - 4s$, provided this expression is non-negative. When $g - 1 = 2s$ the moduli space is again a K3 surface, isogenous to S [Muk87].

Let $M_{2,\omega}(D)$ denote the moduli space of semistable rank-two vector bundles on D with canonical determinant. Consider the non-abelian Brill-Noether loci defined by Mukai [Muk01]

$$M_{2,\omega,s}(D) := \{E \in M_{2,\omega}(D) : h^0(E) \geq 2 + s\},$$

which has expected dimension $3g - 3 - \binom{s+3}{2}$.

Restricting bundles from S to D yields examples of non-abelian Brill-Noether loci with dimensions frequently larger than expected. Mukai has developed a program, supported by beautiful examples, that seeks to characterize $\mathcal{K}_g \subset \mathcal{M}_g$ in terms of special non-abelian Brill-Noether loci.

A particularly striking result along these lines is:

Theorem 1. [Muk96, Thm. 1] *Let D be a generic curve of genus eleven. Then there exist a genus eleven K3 surface (S, h) and an embedding $D \hookrightarrow S$, which are unique up to isomorphisms. Furthermore, we can characterize S as $M_{2,h,5}(T)$, where $T = M_{2,\omega,5}(D)$, which is also a genus eleven K3 surface.*

Thus $\varphi_{11} : \mathcal{P}_{11} \to \mathcal{M}_{11}$ is birational and the K3 surface can be recovered via moduli spaces of vector bundles on D. The theorem remains true for the generic hexagonal curves of genus eleven; see [Muk96, Thm. 3]. Hexagonal curves form a divisor in $\mathcal{M}_{11}$, so φ_{11} remains an isomorphism over the generic point of the hexagonal locus. We will analyze what happens over the pentagonal locus.

We shall need a similar result along these lines in genus seven [Muk95]. Let $\mathrm{OG}(5, 10) \subset \mathbb{P}^{15}$ denote the orthogonal Grassmannian, parametrizing five dimensional isotropic subspaces for a ten dimensional non-degenerate quadratic form. Let G denote the corresponding orthogonal group. The intersection of $\mathrm{OG}(5, 10)$ with a generic six dimensional subspace $\Pi_6 \subset \mathbb{P}^{15}$ is a canonical curve of genus seven; its intersection with a seven dimensional subspace $\Pi_7 \subset \mathbb{P}^{15}$ is a K3 surface of genus seven. Thus we obtain rational maps

$$\begin{array}{rcl} \mathrm{Gr}(7, 16)/G & \dashrightarrow & \mathcal{M}_7 \\ \mathrm{Gr}(8, 16)/G & \dashrightarrow & \mathcal{F}_7 \\ \mathrm{Fl}(7, 8, 16)/G & \dashrightarrow & \mathcal{P}_7 \end{array}$$

where $\mathrm{Fl}(7,8,16) \subset \mathrm{Gr}(7,16) \times \mathrm{Gr}(8,16)$ is the flag variety. Mukai shows that each of these is birational.

3. Degeneration of Mukai's construction over the pentagonal locus

Let $V \subset \mathbb{P}^6$ denote the Fano threefold of index two obtained as a generic linear section of the Grassmannian $\mathrm{Gr}(2,5)$.

Proposition 2. *Let D be a pentagonal curve of genus eleven, i.e., D admits a basepoint free g^1_5. Assume D is generic. Consider*

$$D \hookrightarrow \mathbb{P}^6,$$

the embedding induced by the linear series adjoint to the g^1_5. This admits a canonical factorization

$$D \subset V \subset \mathbb{P}^6.$$

Proof. Let $\phi_1 : D \to \mathbb{P}^1$ be the degree five morphism. Write $(\phi_1)_*\mathcal{O}_D = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{F}$ using the trace homomorphism; by relative duality, we have

$$(\phi_1)_*\omega_D = \omega_{\mathbb{P}^1} \oplus \mathcal{F}^\vee \otimes \omega_{\mathbb{P}^1}$$

and

$$\Gamma(D, \omega_D) = \Gamma(\mathbb{P}^1, \omega_{\mathbb{P}^1} \oplus \mathcal{F}^\vee \otimes \omega_{\mathbb{P}^1}).$$

Moreover $\mathcal{F}^\vee \otimes \omega_{\mathbb{P}^1}$ is globally generated, as the divisors in the basepoint free g^1_5 impose four conditions on the canonical series. Hence the canonical embedding factors

$$D \hookrightarrow \mathbb{P}(\mathcal{F} \otimes T_{\mathbb{P}^1}) \to \mathbb{P}^{10}.$$

Our genericity assumption is that the vector bundles above are as 'balanced' as possible, i.e.,

$$\mathcal{E} := \mathcal{F} \otimes T_{\mathbb{P}^1} \simeq \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-2)^{\oplus 3}.$$

Since $\mathcal{E}^\vee \otimes \mathcal{O}_{\mathbb{P}^1}(-1)$ is still globally generated, we also have a factorization of the adjoint morphism

$$D \hookrightarrow \mathbb{P}(\mathcal{E} \otimes \mathcal{O}_{\mathbb{P}^1}(1)) \to \mathbb{P}^6;$$

the image of the projective bundle is a cone over the Segre threefold $\mathbb{P}^1 \times \mathbb{P}^2$.

The tautological exact sequence on $p : \mathbb{P}(\mathcal{E}) \to \mathbb{P}^1$ takes the form

$$0 \to \mathcal{O}_{\mathbb{P}(\mathcal{E})}(-1) \to p^*\mathcal{E} \to Q \to 0;$$

twisting yields

$$0 \to \mathcal{O}_{\mathbb{P}(\mathcal{E})}(-1) \otimes p^*\mathcal{O}_{\mathbb{P}^1}(2) \to p^*(\mathcal{E} \otimes \mathcal{O}_{\mathbb{P}^1}(2)) \to Q' := Q \otimes p^*\mathcal{O}_{\mathbb{P}^1}(2) \to 0.$$

Note that $\mathrm{rank}(Q') = 3$, $h^0(Q') = 5$, Q' is globally generated, and $c_1(Q') = c_1(\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)) - p^*c_1(\mathcal{O}_{\mathbb{P}^1}(1))$. Restricting to $D \subset \mathbb{P}(\mathcal{E})$ gives

$$c_1(Q'|D) = K_D - g^1_5,$$

the adjoint divisor. The classifying map for $Q'|D$ gives a morphism

$$D \to \mathrm{Gr}(2,5)$$

factoring through a codimension three linear section, which is V. $\square$

Remark 3. We isolate where the generality assumption is used: It is necessary that $\mathcal{E}$ not admit any summands of degree ≤ -3, or equivalently,

$$(\phi_1)_*\omega_D = \omega_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(2)^{\oplus 3}.$$

Theorem 4. *Let D be a generic pentagonal curve of genus eleven and $(\phi_1, \phi_2) : D \to \mathbb{P}^1 \times V$ the embedding given by the degree five covering and Proposition 2. We have*

- *$\varphi_{11}^{-1}(D) \simeq \mathbb{P}^2$, specifically, the K3 surfaces containing D are the codimension-two linear sections of $\mathbb{P}^1 \times V$ containing D.*
- *Fix two distinct points $d_1, d_2 \in D$ with $\phi_1(d_1) = \phi_1(d_2) =: p$; given a conic Z satisfying*
$$d_1, d_2 \in Z \subset \phi_1^{-1}(p),$$
there exists a unique K3 surface S containing D and Z.

The K3 surface S has lattice polarization

(1)
$$\begin{array}{c|ccc} & h & E & Z \\ \hline h & 20 & 5 & 2 \\ E & 5 & 0 & 0 \\ Z & 2 & 0 & -2 \end{array}$$

where E is the elliptic fibration inducing the g^1_5 on $D \in |h|$.

Proof. We present the basic geometric set-up. We have

$$\mathbb{P}^1 \times V \subset \mathbb{P}^1 \times \mathbb{P}^6 \subset \mathbb{P}^{13}$$

where the last inclusion is the Segre embedding. Given a flag

$$\mathbb{P}^{10} \subset \mathbb{P}^{11} \subset \mathbb{P}^{13},$$

intersecting with $\mathbb{P}^1 \times V$ yields

$$D \subset S \subset \mathbb{P}^1 \times V,$$

where D is a canonically embedded pentagonal curve of genus five, and S is a K3 surface with lattice polarization:

$$\begin{array}{c|cc} & h & E \\ \hline h & 20 & 5 \\ E & 5 & 0 \end{array}$$

Fixing D, the K3 surfaces S containing D correspond to the $\mathbb{P}^{11}$'s in $\mathbb{P}^{13}$ containing the fixed $\mathbb{P}^{10}$, which are parametrized by $\mathbb{P}^2$. This proves the first assertion.

We prove the existence assertion of the second part. The assumption $\phi_1(d_1) = \phi_1(d_2) =: p$ means that

$$d_1, d_2 \in \{p\} \times V \subset \mathbb{P}^1 \times V.$$

Two-dimensional linear sections S containing D and the desired conic correspond to conics $Z \subset V$ passing through $d_1, d_2 \in V$. Indeed, S may be recovered from Z:

$$S = \mathrm{span}(D \cup_{d_1,d_2} Z) \cap (\mathbb{P}^1 \times V).$$

Note that each such S is automatically regular along D. For generic pentagonal D and $p \in \mathbb{P}^1$, S must be regular everywhere. We see this by a parameter count.

Pairs (S,D) where S is a K3 surface with lattice polarization (1) and $D \in |h|$ depend on $17+11=28$ parameters. Pairs (D,p) where D admits a degree-five morphism to $\mathbb{P}^1$ and $p \in \mathbb{P}^1$ depend on $30-3+1=28$ parameters. Thus the (D,p) arising from singular K3 surfaces cannot be generic.

The uniqueness assertion boils down to an enumerative problem: How many conics pass through two prescribed generic points of V? Recall that the Fano threefold V may be obtained from a smooth quadric hypersurface $Q \subset \mathbb{P}^4$ explicitly (see e.g. [IŠ79, p. 173]):

$$\begin{tikzcd}[column sep=small] & \mathrm{Bl}_\ell(V) = \mathrm{Bl}_m(Q) \arrow[dl] \arrow[dr] & \\ V & & Q \end{tikzcd}$$

The rational map $V \dashrightarrow Q$ arises from projecting from a line $\ell \subset V$; the inverse $Q \dashrightarrow V$ is induced by the linear series of quadrics vanishing along a twisted cubic curve $m \subset Q$. Note that V contains a two-parameter family of lines which sweep out the threefold, so we may take ℓ disjoint from d_1 and d_2.

Using this modification, our enumerative problem may be transferred to Q: How many conics in Q pass through two prescribed generic points $d_1, d_2 \in Q$ and meet a twisted cubic $m \subset Q$ twice at unprescribed points? There is one such curve. Indeed, projection from the line spanned by the two prescribed points gives a morphism

$$m \to \mathbb{P}^2$$

with image a nodal cubic. Let $m_1, m_2 \in m$ be the points lying over the node; the set $\{m_1, m_2, d_1, d_2\} \subset Q \subset \mathbb{P}^4$ lies in a plane P. The stipulated conic arises as the intersection $P \cap Q$. □

4. Specializing pentagonal curves of genus 11

For our ultimate application to genus 7 curves, we must specialize the construction of Section 3 further:

Theorem 5. *Let C_1 denote a generic curve of genus five, $c_1, \ldots, c_7 \in C_1$ generic points, and $\psi : C_1 \to \mathbb{P}^1 =: R'$ a degree four morphism. Let $D = C_1 \cup_{c_j = \psi(c_j)} R'$ denote the genus eleven nodal curve obtained by gluing C_1 and R', which is automatically pentagonal. Fix an additional generic point $c_0 \in C_1$, and write $\psi^{-1}(\psi(c_0)) = \{c_0, c_1', c_2', c_3'\}$. Then there exists a unique embedding $D \hookrightarrow S$ where S is a K3 surface with lattice polarization*

	f	C_1	C_2	R'
f	4	7	1	6
C_1	7	8	3	7
C_2	1	3	-2	0
R'	6	7	0	-2

where $C_2 \simeq \mathbb{P}^1$ intersects C_1 at $\{c_1', c_2', c_3'\}$.

Proof. We claim that the argument of Theorem 4 applies, as D satisfies Remark 3. Indeed, it suffices to show that

$$h^0(\omega_D(-2g^1_5)) = 3;$$

if $(\phi_1)_*\omega_D$ failed to have the expected decomposition then

$$h^0(\mathbb{P}^1, (\phi_1)_*\omega_D) > 3,$$

a contradiction.

If $h^0(\omega_D(-2g^1_5)) > 3$ then

$$m := h^0(\mathcal{O}_D(2g^1_5)) = h^1(\mathcal{O}_D(2g^1_5)) > 3;$$

clearly $2g^1_5$ is basepoint free, so we have a morphism

$$D \to \mathbb{P}^{m-1}, \quad m \geq 4.$$

The image of R' under this morphism is a plane conic, hence the images of $c_1, \dots, c_7$ are distinct coplanar points. This means that on C_1

$$2g^1_4 - c_1 - \dots - c_7, \quad g^1_4 = g^1_5|C_1$$

is effective, contradicting the genericity of $c_1, \dots, c_7$.

Now we take $d_1 = c_0$ and $d_2 = \psi(c_0) \in R' \simeq \mathbb{P}^1$. Thus D is contained in a distinguished surface S containing a rational curve $Z \ni d_1, d_2$. Hence S has lattice polarization

$$\begin{array}{c|cccc} & C_1 & R' & E & Z \\ \hline C_1 & 8 & 7 & 4 & 1 \\ R' & 7 & -2 & 1 & 1 \\ E & 4 & 1 & 0 & 0 \\ Z & 1 & 1 & 0 & -2 \end{array}$$

containing the intersection matrix (1). Using the identifications

$$h = C_1 + R',\ E = C_1 + C_2 - f,\ Z = C_1 - f,\ R' = R',$$

we obtain the desired lattice polarization.

It remains to show that for generic inputs the surface S is in fact smooth. As before, this follows from a parameter count. The data

$$(C_1, c_0, c_1, \dots, c_7)$$

consists of a genus five curve (12 parameters), a choice of g^1_4 on that curve (1 parameter), and eight generic points, for a total of 21 parameters. On the output side, we have a K3 surface with the prescribed lattice polarization of rank four (16 parameters) and a curve in the linear series $|C_1|$ (5 parameters). Thus for generic input data, the resulting surface is necessarily smooth. □

5. Genus 7 K3 surfaces and rational normal septic curves

Consider the moduli space of lattice-polarized K3 surfaces of type

$$\Lambda' := \begin{array}{c|cc} & C & R' \\ \hline C & 12 & 7 \\ R' & 7 & -2 \end{array}$$

and let $\mathcal{P}_\Lambda$ denote the moduli space of pairs (S, D), where S is such a K3 surface and $D \in |C|$ is smooth and meets R' transversely.

Proposition 6. *The forgetting morphism*

$$\varphi_{\Lambda'} : \mathcal{P}_{\Lambda'} \to \mathcal{M}_{7;7}$$

is generically finite.

Note that the varieties are both of dimension 25. The main ingredient of the proof is:

Lemma 7. *Let* $\mathcal{M}_{0,7}(\mathrm{OG}(5,10),7)$ *be the moduli space of pointed mappings of degree* 7ℓ*, where* $\ell \in H_2(\mathrm{OG}(5,10),\mathbb{Z})$ *is Poincaré dual to the hyperplane class* h*. Then the evaluation map*

$$\mathrm{ev}^7 : \mathcal{M}_{0,7}(\mathrm{OG}(5,10)),7) \to \mathrm{OG}(5,10)^7$$

is dominant.

Proof. Recall that $\dim(\mathrm{OG}(5,10)) = 10$ and the canonical class is

$$K_{\mathrm{OG}(5,10)} = -8h.$$

The expected dimension of the moduli space is 70, so we expect ev^7 to be generically finite.

By [dJHS11, 15.7], the evaluation map

$$\mathrm{ev} : \mathcal{M}_{0,7}(\mathrm{OG}(5,10)),1) \to \mathrm{OG}(5,10)$$

is surjective, i.e., pointed lines dominate $\mathrm{OG}(5,10)$. It follows from generic smoothness that $H^1(N_{\ell/\mathrm{OG}(5,10)}(-1)) = 0$ [AK03, p.33] and

$$N_{\ell/\mathrm{OG}(5,10)} \simeq \mathcal{O}_{\mathbb{P}^1}^{\oplus 3} \oplus \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus 6}.$$

Fix a generic chain of seven lines in $\mathrm{OG}(5,10)$

$$\mathcal{C}_0 := \ell_1 \cup_{e_{12}} \ell_2 \cup_{e_{23}} \ldots \cup_{e_{67}} \ell_7,$$

as well as generic points $r_j(0) \in \ell_j$. Consider the following moduli problem: Fix a smooth base scheme B with basepoint 0 and morphisms $r_j : B \to \mathrm{OG}(5,10), j = 1,\ldots,7$ mapping 0 to the point $r_j(0)$ specified above. We are interested in subschemes

$$\begin{array}{ccccc} r_1,\ldots,r_7 & \subset & \mathcal{C} & \subset & \mathrm{OG}(5,10)\times B \\ & \searrow & \downarrow & \swarrow & \\ & & B & & \end{array}$$

all flat over B, with the distinguished fiber of $\mathcal{C}$ equal to $\mathcal{C}_0$. The deformation theory of this Hilbert scheme as a scheme over B [AK03, §6] is governed by the tangent space

$$\Gamma(N_{\mathcal{C}_0/\mathrm{OG}(5,10)}(-r_1(0) - \cdots - r_7(0)))$$

and the obstruction space

$$H^1(N_{\mathcal{C}_0/\mathrm{OG}(5,10)}(-r_1(0) - \cdots - r_7(0))).$$

If the latter group is 0, the Hilbert scheme is flat over B of the expected dimension

$$\chi(N_{\mathcal{C}_0/\mathrm{OG}(5,10)}(-r_1(0) - \cdots - r_7(0))) = 0.$$

Since $\mathcal{C}_0$ is a chain of rational curves, it suffices to exclude higher cohomology on each irreducible component ℓ_i. For $i = 1,7$ we have

$$N_{\mathcal{C}_0/\mathrm{OG}(5,10)}|\ell_i = \mathcal{O}_{\mathbb{P}^1}^3 \oplus \mathcal{O}_{\mathbb{P}^1}(1)^5 \oplus \mathcal{O}_{\mathbb{P}^1}(2)$$

which has trivial higher cohomology, even after twisting by $\mathcal{O}_{\ell_i}(-r_i(0))$. For $i = 2, \ldots, 6$, we have

$$N_{\mathcal{C}_0/\mathrm{OG}(5,10)}|\ell_i = \mathcal{O}_{\mathbb{P}^1}^3 \oplus \mathcal{O}_{\mathbb{P}^1}(1)^4 \oplus \mathcal{O}_{\mathbb{P}^1}(2)^2,$$

reflecting the two attaching points on ℓ_i. Twisting by $\mathcal{O}_{\ell_i}(-r_i(0))$, we have no higher cohomology as well.

Choosing $r_j : B \to \mathrm{OG}(5,10)$ suitably generic, we conclude there is a zero dimensional collection of rational septic curves passing through seven generic points of $\mathrm{OG}(5,10)$. □

We complete the proof of Proposition 6. Let D denote a generic curve of genus seven and $r_1, \ldots, r_7 \in D$ generic points. As we recalled in Section 2, C' arises as a linear section of $\mathrm{OG}(5,10)$, and Lemma 7 yields a septic rational curve $R' \subset \mathrm{OG}(5,10)$ containing these seven points. The intersection

$$\mathrm{OG}(5,10) \cap \mathrm{span}(D \cup R')$$

is a K3 surface, with the prescribed lattice polarization.

Remark 8. What is the degree of $\varphi_{\Lambda'}$? The birationality results quoted in Section 2 imply that the degree equals the degree of the generically-finite mapping

$$\mathrm{ev}^7 : \mathcal{M}_{0,7}(\mathrm{OG}(5,10)), 7) \to \mathrm{OG}(5,10)^7.$$

See the Appendix for a proof that $\deg(\mathrm{ev}^7) = 71$.

6. From genus 11 to genus 7

Theorem 9. *Let $\pi : C \to R' = \mathbb{P}^1$ be a tetragonal curve of genus seven, $C \subset \mathbb{P}^3$ the adjoint embedding as a curve of degree eight, $c_1, \ldots, c_7 \in C$ generic points. Then there exists a unique embedding*

$$\varpi : R' \hookrightarrow \mathbb{P}^3, \quad \varpi(\pi(c_j)) = c_j$$

such that there exists a quartic surface S containing both C and R'.

The K3 surfaces in this case have lattice polarization:

$$(2) \qquad \Lambda = \begin{array}{c|ccc} & f & C & R' \\ \hline f & 4 & 8 & 6 \\ C & 8 & 12 & 7 \\ R' & 6 & 7 & -2 \end{array}$$

Proof. We regard Theorem 5 as a special case of this, via the specialization

$$C \rightsquigarrow C_1 \cup C_2.$$

We can deform the K3 surfaces in Theorem 5 to the K3 surfaces with lattice polarization (2). Under this deformation, R' deforms to a rational curve in the nearby fibers; smooth rational curves in K3 surfaces always deform provided their divisor classes remain algebraic.

We claim that $C_1 \cup C_2$ deforms to a generic tetragonal curve of genus seven, with seven generic marked points traced out by R'.

For our purposes, we would like to restrict the curve C to be tetragonal. This is equivalent (see [GL87]) to imposing a lattice polarization of the type

$$\begin{array}{c|ccc} & C & R' & E \\ \hline C & 12 & 7 & 4 \\ R' & 7 & -2 & a \\ E & 4 & a & 0 \end{array}$$

where E is the class of a fiber of an elliptic fibration. Restricting $\varphi_{\Lambda'}$ to each such lattice polarization, we obtain a generically finite morphism to the Brill-Noether divisor $\mathcal{T} \subset \mathcal{M}_{7;7}$ corresponding to the tetragonal curves. In our geometric analysis, it will be convenient to use a different basis

$$\begin{array}{c|ccc} & C & R' & f \\ \hline C & 12 & 7 & 8 \\ R' & 7 & -2 & 7-a \\ f & 8 & 7-a & 4 \end{array}$$

where $f = C - E$. We will restrict our attention to the particular component with $a = 1$, i.e., R' is a *section* of the elliptic fibration inducing the g^1_4 on C. This is $\mathcal{P}_\Lambda$, where Λ is defined in (2).

Theorem 9 asserts that the morphism

$$\varphi_\Lambda : \mathcal{P}_\Lambda \to \mathcal{T}$$

has degree one. Consider the specialization $C \rightsquigarrow C_1 \cup C_2$ as above. After specialization, Theorem 5 guarantees a *unique* K3 surface containing $C_1 \cup C_2$. Thus φ_Λ has degree at least one. If the degree were greater than one, then a generic point of $\mathcal{T}$ would yield at least *two* surfaces S and S', with specializations S_0 and S'_0. These are necessarily K3 surfaces, by the following result about degenerate quartic surfaces:

Lemma 10. *Consider the projective space $\mathbb{P}^{34}$ parametrizing all quartic surfaces in $\mathbb{P}^3$. Let $\Sigma \subset \mathbb{P}^{34}$ denote the surfaces with singularities worse than ADE singularities. Then the codimension of Σ is at least four.*

Proof. We first address the non-isolated case. The reducible surfaces of this type—unions of two quadric surfaces or a plane and a cubic—have codimension much larger than four. The irreducible surfaces are classified by Urabe [Ura86, §2], building on the work of numerous predecessors over the last century. In each case, the codimension is at least four.

There is also a substantial literature on the classification of isolated singularities of quartic surfaces, e.g., [Sha81, Deg89, IN04]. However, it will be more convenient for us to give a direct argument, rather than refer to the details of a complete classification.

Since $\mathbb{P}^3$ is homogeneous, it suffices to show that the surfaces with a non ADE singularity at $p = [0, 0, 0, 1]$ have codimension ≥ 3 in the locus of surfaces singular at p. The equations with multiplicity > 2 at p have codimension ≥ 6 and thus can be ignored; the equations of multiplicity two having non-reduced tangent cone at p have codimension 3. The equations with an isolated singularity of multiplicity two at p (and reduced tangent cone) are of ADE type. □

It only remains to exclude ramification at the generic point of the stratum, i.e., that the fibers of φ_Λ have zero dimensional tangent space. Section 7 is devoted to proving this. $\square$

We see these results as special cases of a more general statement of Mukai [Muk96, §10]:

$\varphi_{13} : \mathcal{P}_{13} \to \mathcal{K}_{13}$ is birational onto its image.

This might be approached via a degeneration argument as follows. Consider the stable curve $B = C_1 \cup C_2 \cup R'$ as above, i.e.,

- $\psi : C_1 \to \mathbb{P}^1$ is tetragonal of genus five;
- $R' \simeq \mathbb{P}^1$ and is glued to C_1 at seven generic points via ψ;
- $C_2 \simeq \mathbb{P}^1$, is disjoint from R', and meets C_1 in three points on a generic fiber of ψ.

Note that B has arithmetic genus thirteen and is contained in the image of φ_{13}, or more precisely, a suitable extension of φ_{13} which we now describe.

Let $\overline{\mathcal{P}}_g$ denote the irreducible component moduli stack of stable log pairs (S, C) containing $\mathcal{P}_g$. There is an extension

$$\varphi_g : \overline{\mathcal{P}}_g \to \overline{\mathcal{M}}_g$$

to the moduli space of stable curves. Suppose there is a point $C_\eta \in \mathcal{M}_g$ such that $C_\eta \subset S_\eta, S'_\eta$, both K3 surfaces of genus g. Suppose there is a specialization $C_\eta \rightsquigarrow C_0$ admitting a specialization $S_\eta \rightsquigarrow S_0$ to a K3 surface containing C_0. Then there exists a specialization $S'_\eta \rightsquigarrow S'_0$ to a K3 surface containing C_0. In other words, the part of $\overline{\mathcal{P}}_g$ arising from K3 surfaces has no 'holes'. The reason to expect this is that the K3 surfaces S_η and S'_η ought to be isogenous, and this isogeny should also specialize, so one cannot become singular without the other becoming singular.

This would require additional argument, e.g., by identifying a situation where we can analyze *a priori* the fibers of φ_g. This might be possible with a suitable GIT construction of $\mathcal{P}_{13}$.

7. Deformation computations

7.1. **Generalities.** We review the formalism of deformations of pairs, following [Kaw78]. For the moment, S denotes a smooth projective variety and $D \subset S$ a reduced normal crossings divisor. We have exact sequences

$$0 \to T_S(-D) \to T_S\langle -D\rangle \to T_D \to 0$$

and

$$0 \to T_S\langle -D\rangle \to T_S \to N_{D/S} \to 0,$$

where $T_S\langle -D\rangle$ means vector fields on S with logarithmic zeros along D. The tangent space to the deformation space of (S, D) is given by $H^1(T_S\langle -D\rangle)$. The sequence

$$\Gamma(N_{D/S}) \to H^1(T_S\langle -D\rangle) \to H^1(T_S)$$

may be interpreted 'first-order deformations of (S, D) leaving S unchanged arise from deformations of $D \subset S$'; the sequence

$$H^1(T_S(-D)) \to H^1(T_S\langle -D\rangle) \to H^1(T_D)$$

means that we may interpret $H^1(T_S(-D))$ as 'first-order deformations of (S,D) leaving D unchanged'.

Our situation is a bit more complicated as our boundary consists of *two* log divisors deforming independently. If $D=\{x=0\}$ and $R'=\{y=0\}$ meet transversally at $x=y=0$ then $T_S\langle -D\rangle\langle -R'\rangle$ is freely generated by $x\frac{\partial}{\partial x}$ and $y\frac{\partial}{\partial y}$. First-order deformations of (S,D,R') are given by $H^1(T_S\langle -D\rangle\langle -R'\rangle)$.

We analyze the ramification of the forgetting morphism φ from the deformation space of (S,D,R) to the deformation space of $(D,D\cap R')$. A slight variation on one of the standard exact sequences above

$$0\to T_S\langle -R\rangle(-D)\to T_S\langle -D\rangle\langle -R'\rangle\to T_D\langle -R'\rangle\to 0$$

induces the differential

$$d\varphi: H^1(T_S\langle -D\rangle\langle -R'\rangle)\to H^1(T_D\langle -D\cap R\rangle).$$

Hence the kernel is given by $H^1(T_S\langle -R'\rangle(-D))$, which is zero precisely when ϕ is unramified. The exact sequence

$$0\to T_S\langle -R'\rangle(-D)\to T_S(-D)\to N_{R'/S}(-D\cap R')\to 0$$

induces

$$\begin{aligned} &\Gamma(N_{R'/S}(-D\cap R'))\to\\ H^1(T_S\langle -R'\rangle(-D))\to H^1(T_S(-D))&\xrightarrow{h} H^1(N_{R'/S}(-D\cap R')).\end{aligned}$$

Thus φ is unramified if $\Gamma(N_{R'/S}(-D\cap R'))=0$ and h is injective.

7.2. **Our specific situation.** We specify what the various objects are: First,

$$S=\{F_1=F_2=0\}\subset\mathbb{P}^1\times\mathbb{P}^3.$$

is a smooth complete intersection of forms of bidegree $(1,2)$. The divisor

$$D=\{L=0\}\subset S$$

is a hyperplane section, i.e., L is of bidegree $(1,1)$. The divisor $R'\subset S$ is a smooth rational curve of bidegree $(1,6)$, meeting D transversally, in seven points.

Since $\deg(N_{R'/S}(-D\cap R'))=-2-7$ it has no global sections; the higher cohomology is computed via Serre duality

$$H^1(N_{R'/S}(-D\cap R')=\Gamma(\mathcal{O}_{R'}(D\cap R'))^\vee=\Gamma(\mathcal{O}_{\mathbb{P}^1}(7))^\vee.$$

We claim that $H^1(T_S(-D))$ is eight dimensional.

Remark 11. $H^1(T_S(-D))$ is therefore eight dimensional when S is a generic K3 surface of degree 12. This has a nice global interpretation via the Mukai construction [Muk96]: D (resp. S) is a codimension nine (resp. eight) linear section of the orthogonal Grassmannian $\mathrm{OG}(5,10)$, so the K3 surfaces containing a fixed D are parametrized by $\mathbb{P}^8$.

To prove the claim, use the normal bundle exact sequence

$$0\to T_S(-D)\to T_{\mathbb{P}^1\times\mathbb{P}^3}(-1,-1)|S\to N_{S/\mathbb{P}^1\times\mathbb{P}^3}(-1,-1)\to 0,$$

the Koszul complex for $\{F_1,F_2\}$

$$0\to\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^3}(-2,-4)\to\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^3}(-1,-2)^{\oplus 2}\to\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^3}\to\mathcal{O}_S\to 0$$

and its twist by $T_{\mathbb{P}^1\times\mathbb{P}^3}(-1,-1)$. Applying the Kunneth formula to compute the cohomologies of the twists of $T_{\mathbb{P}^1\times\mathbb{P}^3}$, we find that

$$\Gamma(T_{\mathbb{P}^1\times\mathbb{P}^3}(-1,-1)|S) = H^1(T_{\mathbb{P}^1\times\mathbb{P}^3}(-1,-1)|S) = 0.$$

We also find that

$$\Gamma(N_{S/\mathbb{P}^1\times\mathbb{P}^3}(-1,-1)) = \Gamma(\mathcal{O}_S(0,1))^{\oplus 2} = \Gamma(\mathcal{O}_{\mathbb{P}^3}(1))^{\oplus 2},$$

which is isomorphic to $H^1(T_S(-D))$ by the vanishing above.

In concrete terms, the infinitesimal deformations corresponding to $H^1(T_S(-D))$ take the form

$$S(\epsilon) = \{F_1 + \epsilon L G_1 = F_2 + \epsilon L G_2 = 0\}, \quad G_1, G_2 \in \Gamma(\mathcal{O}_{\mathbb{P}^3}(1)). \tag{3}$$

The exact sequence above therefore takes the form

$$0 \to H^1(T_S\left\langle -R'\right\rangle(-D)) \to \Gamma(\mathcal{O}_{\mathbb{P}^3}(1))^{\oplus 2} \xrightarrow{h} \Gamma(\mathcal{O}_{\mathbb{P}^1}(7)),$$

where h captures the obstructions to deforming the rational curve along an infinitesimal deformation of S. Precisely, suppose we start off with

- $(\mathbb{P}^1, r_1, \ldots, r_7)$ a pointed rational curve;
- a K3 surface $S = \{F_1 = F_2 = 0\} \subset \mathbb{P}^1 \times \mathbb{P}^4$;
- a nodal hyperplane section curve $D \subset S$ with equation $L = 0$;
- a morphism $\iota : \mathbb{P}^1 \to S$ of bidegree $(1,6)$ with $d_j = \iota(r_j) \in D$.

Fix ϵ to be a small parameter, or in algebraic terms, a nilpotent with $\epsilon^2 = 0$. We analyze first-order deformations of

- the K3 surface $S(\epsilon)$ as in (3);
- the morphism $\iota(\epsilon) : \mathbb{P}^1 \to S_\epsilon$ satisfying the constraint

$$d_j = \iota(\epsilon)(r_j). \tag{4}$$

Let $[t,u]$ be coordinates on $\mathbb{P}^1$ and write the seven distinct points $[r_j,1], j = 1,\ldots,7$. Let $P_1,\ldots,P_7 \in k[t,u]_6$ be homogeneous polynomials of degree six such that $P_i(r_j,1) = \delta_{ij}$; these form a basis of $k[t,u]_6$. We factor ι as the composition of the 6-uple embedding

$$\begin{array}{rcl} v : \mathbb{P}^1 & \to & \mathbb{P}^6 \\ {[t,u]} & \mapsto & [P_1,\ldots,P_7] \end{array}$$

and a linear projection given by the 4×7 matrix

$$D = \begin{pmatrix} d_{01} & d_{02} & \ldots & d_{07} \\ d_{11} & d_{12} & \ldots & d_{17} \\ d_{21} & d_{22} & \ldots & d_{27} \\ d_{31} & d_{32} & \ldots & d_{37} \end{pmatrix},$$

whose jth column consists of the coordinates of d_j. The condition that $\iota(\mathbb{P}^1) \subset S$ translates into

$$F_m(t,u; \sum_j d_{0j}P_j, \ldots, \sum_j d_{3j}P_j) = 0 \tag{5}$$

for $m = 1,2$.

The perturbation of the matrix D entails rescaling each column of D by a multiplicative scalar, to first order. This takes the form

$$D(\epsilon) = (d_{ij}(1 + s_j\epsilon)),$$

keeping in mind that the case $s_1 = s_2 = \ldots = s_7$ induces a trivial deformation, i.e., rescaling D by a constant. The condition (4) is automatically satisfied.

We analyze the condition

$$\iota(\epsilon)(\mathbb{P}^1) \subset S(\epsilon)$$

to first order. This can be written

$$\begin{array}{l} F_m(t,u;\sum_j d_{0j}(1+\epsilon s_j)P_j,\ldots)+ \\ \epsilon(L\cdot G_m)(t,u;\sum_j d_{0j}(1+\epsilon s_j)P_j,\ldots)=0 \end{array}$$

for $m = 1, 2$. Writing x_0, x_1, x_2, x_3 for the homogeneous coordinates on $\mathbb{P}^3$ and extracting the first derivative—the constant term in ϵ is zero by (5)—we obtain

$$\sum_{k=0}^{3} \frac{\partial F_m}{\partial x_k} \cdot \sum_j d_{0j} s_j P_j + L \cdot G_m = 0.$$

Here we should regard $\frac{\partial F_m}{\partial x_k}$ as a homogeneous form of degree 7 in $\{t, u\}$, L also of degree 7, and G_i of degree 6. Thus we obtain two homogeneous forms of degree 13 in $\{t, u\}$, with each coefficient linear in the variables $s_1, \ldots, s_7$ (and vanishing where $s_1 = \cdots = s_7$) and the eight coefficients of G_1 and G_2. Note, however, that

$$F_m(t, u; d_j) = F_m(t, u; \iota(\epsilon)(r_j)) = 0, \quad j = 1, \ldots, 7,$$

thus $F_m(t, u; \iota(\epsilon)([t, u]))$ is a multiple of $\prod_{i=1,\ldots,7}(t - r_i u)$; the same is true for $L(\iota(\epsilon)([t, u]))$. Dividing out by this, we obtain two homogeneous forms of degree 6 in $\{t, u\}$, with each coefficient linear in the variables; thus we obtain 14 linear equations in 14 independent variables. Again, the equations will vanish if $s_1 = \ldots = s_7$ so these depend on only six parameters.

7.3. **A unirationality result.** To execute the computations outlined in Section 7.2, we must evaluate all the relevant terms in a specific example. In practice, finding such an example is much easier when the underlying parameter spaces are unirational.

Proposition 12. *Consider the Hilbert scheme parametrizing the following data:*

- *points* $r_1, \ldots, r_7 \in \mathbb{P}^1$
- *a rational curve* $C_2 \subset \mathbb{P}^1 \times \mathbb{P}^3$ *of bidegree* $(0, 1)$ *(i.e., a line);*
- *a hyperplane section* $\{L = 0\} \subset \mathbb{P}^1 \times \mathbb{P}^3$ *containing* C_2*;*
- *a morphism* $\iota : \mathbb{P}^1 \to \mathbb{P}^1 \times \mathbb{P}^3$ *of bidegree* $(1, 6)$ *with* $\iota(\{r_1, \ldots, r_7\}) \subset \{L = 0\}$*;*
- *a K3 surface* $S \subset \mathbb{P}^1 \times \mathbb{P}^3$*, given as a complete intersection of forms of degree degree* $(1, 2)$ *containing* C_2 *and* $R' := \iota(\mathbb{P}^1)$*.*

This space is rational.

Below, let f be induced from the hyperplane class of $\mathbb{P}^3$ and $C_1 \cup C_2$ be cut out by $L = 0$. Note that C_1 is residual to C_2 in the hyperplane $D = \{L = 0\}$.

Corollary 13. *Consider the moduli space of lattice polarized K3 surfaces S of type*

$$\begin{array}{c|cccc} & f & C_1 & C_2 & R' \\ \hline f & 4 & 7 & 1 & 6 \\ C_1 & 7 & 8 & 3 & 7 \\ C_2 & 1 & 3 & -2 & 0 \\ R' & 6 & 7 & 0 & -2 \end{array},$$

equipped with an ordering of the points of $C_1 \cap R'$. This space is unirational.

Proof. The construction is step-by-step: Seven ordered points on $\mathbb{P}^1$ are parametrized by a rational variety. The lines C_2 are as well, i.e., the product $\mathbb{P}^1 \times \mathrm{Gr}(2,4)$. For each such line, it is a linear condition for a hypersurface $\{L = 0\}$ to vanish along the line. Once we have C_2 and ι, the K3 surfaces containing $C_2 \cup R'$ are given by the Grassmannian $\mathrm{Gr}(2, I_{C_2 \cup R'}(1,2))$.

Thus we only have to worry about the choice of ι. We interpret this as a section of the projection $\mathbb{P}^1 \times \mathbb{P}^3 \to \mathbb{P}^1$, given by a collection of sextic polynomials

$$[\iota_0(t,u), \dots, \iota_3(t,u)],$$

parametrized by a dense open subset of the projective space $\mathbb{P}^{27}$ on the coefficients. After a linear change of coordinates on $\mathbb{P}^3$, we may assume $L = tx_0 - ux_1$. Then the conditions $\iota(r_j) \in \{L = 0\}$ take the form

$$r_j \iota_0(r_j, 1) - \iota_1(r_j, 1),$$

i.e., seven linear equations on the coefficient space. The resulting parameter space is thus rational. □

7.4. **Concrete example.** We exhibit a specific example over $\mathbb{Q}$ where the morphism φ_Λ is unramified. Its construction closely follows the unirationality proof in Section 7.3.

The relevant data is

- points $r_1 = [0,1], r_2 = [1,1], r_3 = [-1,1], r_4 = [2,1], r_5 = [-2,1], r_6 = [1/2,1], r_7 = [-1/2,1] \in \mathbb{P}^1$;
- a rational curve $C_2 = \{u = x_0 = x_3 = 0\}$;
- a hyperplane section $L = tx_0 - ux_1 \subset \mathbb{P}^1 \times \mathbb{P}^3$, containing C_2;
- a morphism $\iota : \mathbb{P}^1 \to \mathbb{P}^1 \times \mathbb{P}^3$ given by

$$\begin{array}{rcl} x_0 &=& 4t^6 + t^5u + u^6 \\ x_1 &=& t^6 + 21t^5u - 21t^3u^3 + 5tu^5 \\ x_2 &=& t^6 \\ x_3 &=& t^6 + t^5u + t^4u^2 + t^3u^3 + t^2u^4 + tu^5 + u^6; \end{array}$$

- a K3 surface $S \subset \mathbb{P}^1 \times \mathbb{P}^3$, given as a complete intersection

$$\{F_1 = F_2 = 0\}$$

where

$$\begin{aligned} F_1 &= u(-36134306460x_0^2 + 1648259021x_0x_1 + 179920405271x_0x_2 \\ &\quad +72385436466x_0x_3 + 49839426x_1^2 - 3784378416x_1x_2 \\ &\quad -2345703360x_1x_3 - 181391061852x_2^2 - 225811403454x_2x_3 \\ &\quad -36251130006x_3^2) + t(-13678895854x_0^2 + 671675907x_0x_1 \\ &\quad +56417839468x_0x_2 - 8926222977x_0x_3 + 26209164072x_3^2) \end{aligned}$$

and

$$\begin{aligned} F_2 &= u(3638964x_0^2 - 1272831x_0x_1 + 29670963x_0x_2 \\ &\quad -13458270x_0x_3 - 22974x_1^2 + 3555552x_1x_2 \\ &\quad +1904792x_1x_3 - 114701748x_2^2 - 4837990x_2x_3 \\ &\quad +9819306x_3^2) + t(12731586x_0^2 - 575505x_0x_1 \\ &\quad -52280172x_0x_2 - 22071733x_0x_3 + 96004264x_2x_3). \end{aligned}$$

The linear projection matrix is given by

$$D = \begin{pmatrix} 1 & 6 & 4 & 289 & 225 & 35/32 & 33/32 \\ 0 & 6 & -4 & 578 & -450 & 35/64 & -33/64 \\ 0 & 1 & 1 & 64 & 64 & 1/64 & 1/64 \\ 1 & 7 & 1 & 127 & 43 & 127/64 & 43/64 \end{pmatrix}.$$

For the tangent space computation, write

$$\begin{aligned} G_1 &= g_{10}x_0 + g_{11}x_1 + g_{12}x_2 + g_{13}x_3 \\ G_2 &= g_{20}x_0 + g_{21}x_1 + g_{22}x_2 + g_{23}x_3 \end{aligned}$$

and extract the terms linear in ϵ in

$$F_m(t,u;\sum_j d_{0j}(1+\epsilon s_j)P_j,\ldots) + \epsilon(L \cdot G_m)(t,u;\sum_j d_{0j}(1+\epsilon s_j)P_j,\ldots) = 0.$$

This yields two homogeneous forms in s and u of degree 13, with coefficients linear in the g_{ik} and s_j, each divisible by

$$\prod_j (t - r_j u) = t(t-u)(t+u)(t-2u)(t+2u)(t-u/2)(t+u/2).$$

Dividing out, we obtain two forms in s and u of degree 6 with coefficients linear in the g_{ik} and s_j. We will not reproduce these coefficients here, but after Gaussian elimination we are left with the system

$$\begin{gathered} g_{10} = g_{11} = g_{12} = g_{13} = g_{20} = g_{21} = g_{22} = g_{23} = 0 \\ s_1 = s_2 = s_3 = s_4 = s_5 = s_6 = s_7. \end{gathered}$$

Thus the space of infinitesimal deformations of S containing our curves is trivial, hence our morphism φ_Λ is unramified at this point.

8. A related construction

Consider the moduli space $\mathcal{F}_{\Lambda''}$ of lattice-polarized K3 surfaces of type

$$\Lambda'' := \begin{array}{c|cc} & h & R' \\ \hline h & 12 & 5 \\ R' & 5 & -2 \end{array} \tag{6}$$

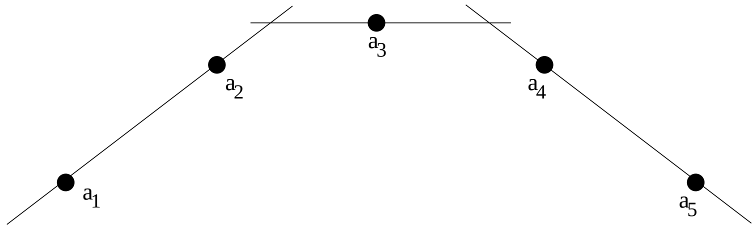

Figure 1. A stable five-pointed curve

and the moduli space $\mathcal{P}_{\Lambda''}$ parametrizing pairs (S, D) with $S \in \mathcal{F}_{\Lambda''}$ and $D \in |h|$ smooth and transverse to R', which has dimension 25. We consider

$$\widehat{\varphi_{\Lambda''}} : \mathcal{P}_{\Lambda''} \to (\mathcal{M}_{7,5} \times \mathcal{M}_{0,5})/\mathfrak{S}_5$$

to a variety of dimension $18 + 5 + 2 = 25$.

Proposition 14. $\widehat{\varphi_{\Lambda''}}$ *is birational.*

Proof. We construct the inverse mapping. Fix $(C, c_1, \ldots, c_5)$ a generic five-pointed curve of genus seven and $(\mathbb{P}^1, a_1, \ldots, a_5)$ a five-pointed curve of genus zero. Realize C as a linear section in $\mathrm{OG}(5, 10)$. We seek a curve R' arising as a rational normal quintic curve $\varpi : \mathbb{P}^1 \to \mathrm{OG}(5, 10)$ with $\varpi(a_i) = c_i, i = 1, \ldots, 5$. The K3 surface S would then arise as the intersection

$$\mathrm{span}(C \cup R') \cap \mathrm{OG}(5, 10).$$

The construction of ϖ boils down to an enumerative computation by Andrew Kresch:

Lemma 15. *Given generic points* $\Lambda_1, \cdots, \Lambda_5 \in \mathrm{OG}(5, 10)$*, and generic points* $a_1, \ldots, a_5 \in \mathbb{P}^1$*, there exists a unique morphism* $\varpi : \mathbb{P}^1 \to \mathrm{OG}(5, 10)$ *with image of degree five such that* $\varpi(a_i) = \Lambda_i$ *for* $i = 1, \ldots, 5$*.*

We have the tautological diagram

$$\begin{array}{ccc} \overline{M}_{0,5}(\mathrm{OG}(5,10),5) & \stackrel{\mathrm{ev}^5}{\to} & \mathrm{OG}(5,10)^5 \\ \phi \downarrow & & \\ \overline{M}_{0,5} & & \end{array}$$

where the ϕ is the forgetting morphism and ev^5 is evaluation at all five marked points. We seek to show that

$$\deg(\mathrm{ev}_1^*[\mathrm{pt}] \cdots \mathrm{ev}_5^*[\mathrm{pt}] \cdot \phi^*(\mathrm{pt})) = 1.$$

We specialize the point in $\overline{M}_{0,5}$ to the configuration of Figure 1.

We assert that

1. there is no line on $\mathrm{OG}(5, 10)$ through two general points
2. the closure of the union of all conics through two general points is a Schubert variety parametrizing spaces containing a particular one-dimensional isotropic vector space Ξ of the ambient ten dimensional space.
3. there is no conic through three general points.

For the first statement, note that the space of lines is parametrized by three dimensional isotropic subspaces of the ambient ten dimensional space. These are contained in the intersection of all the maximal isotropic subspaces parametrized by the line [LM03, Example 4.12]. Thus the space of lines has dimension 15, which is incompatible with any two points containing a line.

For the second statement, consider the conics containing Λ_1 and Λ_2; set $\Xi := \Lambda_1 \cap \Lambda_2$, which is one dimensional, and note that $\Xi^\perp = \Lambda_1 + \Lambda_2$. The $\Lambda \in \mathrm{OG}(5,10)$ satisfying

$$\Xi \subset \Lambda \subset \Xi^\perp$$

are parametrized by $\mathrm{OG}(4,8)$. Recall that $\mathrm{OG}(4,8)$ is just a quadric sixfold by triality; the conics through two generic points are parametrized by the $\mathbb{P}^5$ parametrizing planes containing those two points. Altogether, these trace out the full $\mathrm{OG}(4,8)$ mentioned above.

The last statement follows immediately, as the union of the conics through two points is a proper subvariety of $\mathrm{OG}(5,10)$.

These observations imply that the 'end components' of our stable curve map to conics and the 'middle component' maps to a line. So we are reduced to:

Given

$$\Xi = \Lambda_1 \cap \Lambda_2, \quad \Xi' = \Lambda_4 \cap \Lambda_5$$

how many lines on $\mathrm{OG}(5,10)$ are incident to

- the Schubert variety of $\Lambda \in \mathrm{OG}(5,10)$ satisfying $\Xi \subset \Lambda$;
- the Schubert variety of $\Lambda \in \mathrm{OG}(5,10)$ satisfying $\Xi' \subset \Lambda$;
- the point Λ_3?

We know that lines on $\mathrm{OG}(5,10)$ are in bijection with isotropic three dimensional spaces, so for an isotropic three dimensional W we must have:

(a) W can be extended to a maximal isotropic space containing Ξ;
(b) W can be extended to a maximal isotropic space containing Ξ';
(c) $W \subset \Lambda_3$.

We deduce from (a) that $W \subset \Xi^\perp$ and from (b) that $W \subset (\Xi')^\perp$. It follows that

$$W \subset \Xi^\perp \cap (\Xi')^\perp \cap \Lambda_3.$$

Since Λ_3 is general, this intersection is already three dimensional so that W is uniquely determined and the answer to our enumerative problem is 1. □

References

[AK03] Carolina Araujo and János Kollár. Rational curves on varieties. In *Higher dimensional varieties and rational points (Budapest, 2001)*, volume 12 of *Bolyai Soc. Math. Stud.*, pages 13–68. Springer, Berlin, 2003.

[Deg89] A. I. Degtyarëv. Classification of quartic surfaces that have a nonsimple singular point. *Izv. Akad. Nauk SSSR Ser. Mat.*, 53(6):1269–1290, 1337–1338, 1989.

[dJHS11] A. J. de Jong, Xuhua He, and Jason Michael Starr. Families of rationally simply connected varieties over surfaces and torsors for semisimple groups. *Publ. Math. Inst. Hautes Études Sci.*, (114):1–85, 2011.

[GL87] Mark Green and Robert Lazarsfeld. Special divisors on curves on a $K3$ surface. *Invent. Math.*, 89(2):357–370, 1987.

[HT12] Brendan Hassett and Yuri Tschinkel. Quartic del Pezzo surfaces over function fields of curves, 2012. Preprint.

[IN04] Yuji Ishii and Noboru Nakayama. Classification of normal quartic surfaces with irrational singularities. *J. Math. Soc. Japan*, 56(3):941–965, 2004.

[IŠ79] V. A. Iskovskih and V. V. Šokurov. Biregular theory of Fano 3-folds. In *Algebraic geometry (Proc. Summer Meeting, Univ. Copenhagen, Copenhagen, 1978)*, volume 732 of *Lecture Notes in Math.*, pages 171–182. Springer, Berlin, 1979.

[Kaw78] Yujiro Kawamata. On deformations of compactifiable complex manifolds. *Math. Ann.*, 235(3):247–265, 1978.

[Muk87] S. Mukai. On the moduli space of bundles on $K3$ surfaces. I. In *Vector bundles on algebraic varieties (Bombay, 1984)*, volume 11 of *Tata Inst. Fund. Res. Stud. Math.*, pages 341–413. Tata Inst. Fund. Res., Bombay, 1987.

[Muk95] Shigeru Mukai. Curves and symmetric spaces. I. *Amer. J. Math.*, 117(6):1627–1644, 1995.

[Muk96] Shigeru Mukai. Curves and $K3$ surfaces of genus eleven. In *Moduli of vector bundles (Sanda, 1994; Kyoto, 1994)*, volume 179 of *Lecture Notes in Pure and Appl. Math.*, pages 189–197. Dekker, New York, 1996.

[Muk01] Shigeru Mukai. Non-abelian Brill-Noether theory and Fano 3-folds [translation of Sūgaku **49** (1997), no. 1, 1–24;]. *Sugaku Expositions*, 14(2):125–153, 2001. Sugaku Expositions.

[Sha81] Jayant Shah. Degenerations of $K3$ surfaces of degree 4. *Trans. Amer. Math. Soc.*, 263(2):271–308, 1981.

[Ura86] Tohsuke Urabe. Classification of nonnormal quartic surfaces. *Tokyo J. Math.*, 9(2):265–295, 1986.

Appendix

71 rational septics through 7 general points on OG(5, 10)

Overview and generalities

The orthogonal Grassmannian $\mathrm{OG}(n, 2n)$ is, by definition, one component of the space of n-dimensional subspaces of $V = \mathbb{C}^{2n}$, isotropic for a given nondegenerate symmetric bilinear form on V. By convention we fix the standard bilinear form, for which $\langle v, w\rangle = \sum_{i=1}^{2n} v_i w_{2n+1-i}$, and we take $\mathrm{OG} = \mathrm{OG}(n, 2n)$ to be the component containing $\langle e_1, \dots, e_n\rangle$. Then OG is a homogeneous projective variety of dimension $n(n-1)/2$, with $\deg c_1(\mathrm{OG}) = 2(n-1)$. It follows that the space of rational degree d curves has dimension $n(n-1)/2 - 3 + 2(n-1)d$, so for $n = 5$ and $d = 7$ this is $63 = 7 \cdot 9$ and we expect a finite number of rational septics to pass through 7 general points.

It is known (cf. [FP97, LM11]) that the number of rational curves on OG satisfying incidence conditions imposed by Schubert varieties of codimension ≥ 2 in general position is equal to the corresponding Gromov-Witten invariant:

$$\#\left\{\begin{array}{l}\text{degree } d \text{ rational curves through}\\ \text{general translates of } X_{\lambda^1},\ \dots,\ X_{\lambda^m}\end{array}\right\} = I_d([X_{\lambda^1}], \dots, [X_{\lambda^m}])$$

for $|\lambda^i| \geq 2$, $\sum |\lambda^i| = n(n-1)/2 - 3 + 2(n-1)d + m$, where $I_d([X_{\lambda^1}], \dots, [X_{\lambda^m}])$ denotes the Gromov-Witten invariant

$$\int_{\overline{M}_{0,m}(\mathrm{OG},d)} \mathrm{ev}_1^*[X_{\lambda^1}] \cdots \mathrm{ev}_m^*[X_{\lambda^m}].$$

The space $\overline{M}_{0,m}(\mathrm{OG}, d)$ is Kontsevich's moduli space of stable maps of genus zero m-marked curves to OG in degree d (see [KM94]), and it comes with m evaluation maps (at the marked points) $\mathrm{ev}_1, \dots, \mathrm{ev}_m$ to OG. The Schubert varieties in OG are denoted X_λ, indexed by strict partitions λ whose parts are $< n$. (A partition is called strict if has no repeated parts.) The codimension of X_λ is equal to $|\lambda|$, the sum of the parts of λ. The article [KT04], which includes a determination of the $m = 3$ invariants, gives a geometric description of OG including the Schubert varieties.

Line numbers

The space of lines on OG is known; see [LM03, Example 4.12]. It is itself a projective homogeneous variety, the space $\mathrm{OG}(n-2, 2n)$ of isotropic $(n-2)$-dimensional subspaces of $V = \mathbb{C}^{2n}$. Therefore the computation of the $I_1([X_{\lambda^1}], \dots, [X_{\lambda^m}])$ reduces to the problem of computing intersection numbers on this homogeneous variety.

There is a well-developed theory using divided difference operators on polynomials for performing computations in the cohomology rings of projective homogeneous varieties of linear algebraic groups, due to Bernstein, Gelfand, and Gelfand [BGG73] and Demazure [Dem74]. In the setting of the orthogonal flag variety

OF($2n$), parametrizing a space in OG together with a complete flag of subspaces, this has been worked out explicitly by Billey and Haiman [BH95]. It leads to an explicit formula for the Gromov-Witten invariants counting lines on OG satisfying incidence conditions with respect to Schubert varieties in general position. The formula uses the Schur P-polynomials $P_\lambda = P_\lambda(X)$ indexed by strict partitions λ, which form a $\mathbb{Q}$-basis for the ring $\mathbb{Q}[p_1, p_3, \ldots]$ generated by the odd power sums $p_k = p_k(X) = x_1^k + x_2^k + \ldots$ (cf. Proposition 3.1 of op. cit.). Following op. cit., to these we associate polynomials in $z_1, \ldots, z_n$, which we will denote by $P_\lambda(z_1, \ldots, z_n)$, by sending $p_k(X)$ to $-(1/2)(z_1^k + \cdots + z_n^k)$.

Proposition. *Introduce the divided difference operators on* $\mathbb{Q}[z_1, \ldots, z_n]$*:*

$$\begin{aligned} \partial_i f &= \tfrac{f(z_1,\ldots,z_n) - f(z_1,\ldots,z_{i+1},z_i,\ldots,z_n)}{z_i - z_{i+1}}, \\ \partial_{\hat{1}} f &= \tfrac{f(z_1,\ldots,z_n) - f(-z_2,-z_1,z_3,\ldots,z_n)}{-z_1 - z_2}, \end{aligned}$$

and for $i \le j$ let $\partial_{i\ldots j}$ denote $\partial_i \partial_{i+1} \ldots \partial_j$ and let $\partial_{j\ldots i}$ denote $\partial_j \ldots \partial_i$. Then for any m and $\lambda^1, \ldots, \lambda^m$ satisfying $|\lambda^i| \ge 2$, $\sum |\lambda^i| = n(n-1)/2 - 3 + 2(n-1) + m$, if we set

$$F = \prod_{i=1}^{m} \partial_{\hat{1}} P_{\lambda^i}(z_1, \ldots, z_n)$$

with the above convention on P-polynomials, then we have

$$I_1([X_{\lambda^1}], \ldots, [X_{\lambda^m}]) = \begin{cases} \partial_{2\ldots n-1}\partial_{1\ldots n-2}\partial_{\hat{1}} \cdots \partial_{2\ldots 3}\partial_{1\ldots 2}\partial_{\hat{1}}\partial_{n-2\ldots 1}\partial_{n-1\ldots 2}F & \text{if } n \text{ is even,} \\ \partial_{2\ldots n-1}\partial_{\hat{1}}\partial_{2\ldots n-2}\partial_{1\ldots n-3}\partial_{\hat{1}} \cdots \partial_{2\ldots 3}\partial_{1\ldots 2}\partial_{\hat{1}}\partial_{n-2\ldots 1}\partial_{n-1\ldots 2}F & \text{if } n \text{ is odd,} \end{cases}$$

where the $\cdots$ stand for compositions of operators in which the upper limits of the indices are successively decreased by 2.

Proof. According to Theorem 4 of op. cit., if we work with countably many z variables and follow the above convention for associating a symmetric polynomial in these to a P-polynomial $P_\lambda = P_\lambda(X)$, then $\partial_{\bar{1}} P_\lambda$ represents the cycle class of the space of lines incident to X_λ, and the displayed composition of divided operators sends the polynomial representing the class of a point on the space of lines on OG to 1. So the proposition follows from the observation that the computation may be performed in the polynomial ring $\mathbb{Q}[z_1, \ldots, z_n]$. $\square$

When $n = 5$, there are 1071 Gromov-Witten numbers $I_1([X_{\lambda^1}], \ldots, [X_{\lambda^m}])$, which we take as known in what follows.

Example. *One of these numbers counts the number of lines incident to* 15 *general translates of X_2 (the codimension-*2 *Schubert variety of spaces in* OG *meeting a given isotropic* 3*-dimensional space nontrivially). We have $P_2(X) = p_1^2(X)$ sent to $(1/4)(z_1 + \cdots + z_5)^2$, which upon applying $\partial_{\hat{1}}$ yields $-z_3 - z_4 - z_5$. We evaluate*

$$\partial_2\partial_3\partial_4\partial_{\hat{1}}\partial_2\partial_3\partial_1\partial_2\partial_{\hat{1}}\partial_3\partial_2\partial_1\partial_4\partial_3\partial_2(-z_3 - z_4 - z_5)^{15}$$

and find

$$I_1([X_2], \ldots, [X_2]) = 240240.$$

	τ_4	τ_{31}	τ_{41}	τ_{32}	τ_{42}	τ_{321}	τ_{43}	τ_{421}
τ_1	τ_{41}	$\tau_{41}+\tau_{32}$	τ_{42}	$\tau_{42}+\tau_{321}$	$\tau_{43}+\tau_{421}$	τ_{421}	τ_{431}	τ_{431}
τ_2	τ_{42}	$2\tau_{42}+\tau_{321}$	$\tau_{43}+\tau_{421}$	$\tau_{43}+2\tau_{421}$	$2\tau_{431}$	τ_{431}	τ_{432}	τ_{432}
τ_3	τ_{43}	$\tau_{43}+2\tau_{421}$	τ_{431}	$2\tau_{431}$	τ_{432}	τ_{432}	0	τ_{4321}
τ_{21}	τ_{421}	$\tau_{43}+\tau_{421}$	τ_{431}	τ_{431}	τ_{432}	0	τ_{4321}	0
τ_4	0	τ_{431}	0	τ_{432}	0	τ_{4321}	0	0
τ_{31}	τ_{431}	$2\tau_{431}$	τ_{432}	τ_{432}	τ_{4321}	0	0	0

TABLE 1. Portion of multiplication table for $H^*(\mathrm{OG}(5,10))$

We list a few more such numbers:

$$\begin{aligned}
I_1([X_2],[X_3],[X_{421}],[X_{421}])&=2, & I_1([X_2],[X_{21}],[X_{421}],[X_{421}])&=2,\\
I_1([X_2],[X_{42}],[X_{4321}])&=1, & I_1([X_2],[X_{321}],[X_{4321}])&=0,\\
I_1([X_2],[X_{421}],[X_{432}])&=1, & I_1([X_3],[X_{421}],[X_{431}])&=1,\\
I_1([X_{21}],[X_{421}],[X_{431}])&=1, & I_1([X_4],[X_{421}],[X_{421}])&=0,\\
I_1([X_{31}],[X_{421}],[X_{421}])&=1, & I_1([X_{43}],[X_{4321}])&=1,\\
I_1([X_{421}],[X_{4321}])&=0, & I_1([X_{431}],[X_{432}])&=1.
\end{aligned} \tag{7}$$

CONIC NUMBERS

The *associativity relations* of quantum cohomology (also known as WDVV equations) are a system of polynomial relations in Gromov-Witten invariants, which can be used to deduce new invariants from known ones. We recall the statement, as formulated in [KM94, Eqn. (3.3)], for the case of OG. First, the Poincaré duality involution $\lambda \mapsto \lambda^\vee$ on the set of partitions indexing the Schubert classes of OG (basis of the classical cohomology ring of OG), is such that the set of parts of $\lambda^\vee$ is the complement in $\{1,\dots,n-1\}$ of the set of parts of λ. We have focused on Gromov-Witten invariants involving Schubert classes of codimension ≥ 2 above, because the ones with fundamental or divisor classes reduce to these by the following identities:

$$I_0([X_\lambda],[X_\mu],[X_\nu]) = \int_{\mathrm{OG}} [X_\lambda]\cdot[X_\mu]\cdot[X_\nu],$$
$$I_0([X_{\lambda^1}],\dots,[X_{\lambda^m}]) = 0 \text{ for } m\neq 3,$$

and for $d \geq 1$,

$$I_d([X_{\lambda^1}],\dots,[X_{\lambda^m}],[X_1]) = dI_d([X_{\lambda^1}],\dots,[X_{\lambda^m}]).$$

Since it is needed for the discussion that follows, we record in Table 1 a portion of the multiplication table for the Schubert classes $\tau_\lambda = [X_\lambda]$ in the classical cohomology ring. (One can produce this, e.g., using the Pieri formula of [HB86].)

Given $d \geq 1$, $m \geq 4$ and λ^1, …, λ^m satisfying

$$|\lambda^i| \geq 1, \qquad \sum |\lambda^i| = n(n-1)/2 - 4 + 2d(n-1) + m,$$

the corresponding associativity relation reads

$$\begin{aligned}(8)\qquad &\sum_{d',\mu,A} I_{d'}(\tau_{\lambda^{i_1}},\dots,\tau_{\lambda^{i_a}},\tau_{\lambda^{m-3}},\tau_{\lambda^{m-2}},\tau_\mu)I_{d-d'}(\tau_{\lambda^{j_1}},\dots,\tau_{\lambda^{j_b}},\tau_{\lambda^{m-1}},\tau_{\lambda^m},\tau_{\mu^\vee})\\ &=\sum_{d',\mu,A} I_{d'}(\tau_{\lambda^{i_1}},\dots,\tau_{\lambda^{i_a}},\tau_{\lambda^{m-3}},\tau_{\lambda^{m}},\tau_\mu)I_{d-d'}(\tau_{\lambda^{j_1}},\dots,\tau_{\lambda^{j_b}},\tau_{\lambda^{m-2}},\tau_{\lambda^{m-1}},\tau_{\mu^\vee}),\end{aligned}$$

where the first, respectively second sum is over integers $0 \le d' \le d$, strict partitions μ with parts less than n, and subsets $A \subset \{1,\dots,m-4\}$ such that

$$(9)\qquad \sum_{i\in A\cup\{m-3,m-2\}} |\lambda^i| + |\mu| = n(n-1)/2 + 2d'(n-1) + a,$$

respectively the same condition with $m-2$ replaced by m. In the equations (8)–(9) a, respectively b denotes the cardinality of A, respectively $B := \{1,\dots,m-4\} \smallsetminus A$, and we write $A = \{i_1,\dots,i_a\}$ and $B = \{j_1,\dots,j_b\}$.

In case $d = 2$ in (8) we observe the following: (i) all terms with $d' = 1$, and hence $d - d' = 1$, are known by the previous section; (ii) terms with $d' = 0$ contribute

$$(10)\qquad \sum_\mu \Big(\int_{\mathrm{OG}} \tau_{\lambda^{m-3}}\tau_{\lambda^{m-2}}\tau_{\mu^\vee}\Big) I_2(\tau_{\lambda^1},\dots,\tau_{\lambda^{m-4}},\tau_{\lambda^{m-1}},\tau_{\lambda^m},\tau_\mu)$$

to the left-hand side and

$$(11)\qquad \sum_\mu \Big(\int_{\mathrm{OG}} \tau_{\lambda^{m-3}}\tau_{\lambda^{m}}\tau_{\mu^\vee}\Big) I_2(\tau_{\lambda^1},\dots,\tau_{\lambda^{m-4}},\tau_{\lambda^{m-2}},\tau_{\lambda^{m-1}},\tau_\mu)$$

to the right-hand side; (iii) terms with $d' = 2$ contribute

$$(12)\qquad \sum_\mu \Big(\int_{\mathrm{OG}} \tau_{\lambda^{m-1}}\tau_{\lambda^{m}}\tau_{\mu^\vee}\Big) I_2(\tau_{\lambda^1},\dots,\tau_{\lambda^{m-4}},\tau_{\lambda^{m-3}},\tau_{\lambda^{m-2}},\tau_\mu)$$

to the left-hand side and

$$(13)\qquad \sum_\mu \Big(\int_{\mathrm{OG}} \tau_{\lambda^{m-2}}\tau_{\lambda^{m-1}}\tau_{\mu^\vee}\Big) I_2(\tau_{\lambda^1},\dots,\tau_{\lambda^{m-4}},\tau_{\lambda^{m-3}},\tau_{\lambda^{m}},\tau_\mu)$$

to the right-hand side.

Now it is clear that the associativity relations determine many of the Gromov-Witten numbers $I_2(\tau_{\lambda^1},\dots,\tau_{\lambda^m})$. We spell out the cases of interest, and for each case we will subsequently take the corresponding Gromov-Witten numbers as known. Notice that we always take $d = 2$ in the following applications of (8).

Case 1. Two point conditions: $I_2(\dots,\tau_{4321},\tau_{4321})$. We apply (8) with $\lambda^{m-3} = 1$, $\lambda^{m-2} = 432$, $|\lambda^{m-1}| \ge 2$, $\lambda^m = 4321$. Then (11)–(13) vanish, while (10) contributes $I_2(\tau_{\lambda^1},\dots,\tau_{\lambda^{m-4}},\tau_{\lambda^{m-1}},\tau_{4321},\tau_{4321})$.

Case 2. Point and line conditions: $I_2(\dots,\tau_{432},\tau_{4321})$. We apply (8) with $\lambda^{m-3} = 1$, $\lambda^{m-2} = 431$, $|\lambda^{m-1}| \ge 2$, $\lambda^m = 4321$. Then (11)–(12) vanish, (13) either vanishes or is known by Case 1, and (10) contributes $I_2(\tau_{\lambda^1},\dots,\tau_{\lambda^{m-4}},\tau_{\lambda^{m-1}},\tau_{432},\tau_{4321})$.

Case 3. Point and plane: $I_2(\dots,\tau_{431},\tau_{4321})$. We apply (8) with $\lambda^{m-3} = 1$, $\lambda^{m-2} = 421$, $|\lambda^{m-1}| \ge 2$, $\lambda^m = 4321$. Then (11)–(12) vanish, (13) either vanishes or is known by previous cases, and (10) contributes $I_2(\tau_{\lambda^1},\dots,\tau_{\lambda^{m-4}},\tau_{\lambda^{m-1}},\tau_{431},\tau_{4321})$.

Case 4. Point and X_{421}: $I_2(\dots,\tau_{421},\tau_{4321})$. We apply (8) with $\lambda^{m-3}=1$, $\lambda^{m-2}=321$, $|\lambda^{m-1}|\geq 2$, $\lambda^m=4321$, and proceed as in Case 3.

Case 5. Point and X_{43}: $I_2(\dots,\tau_{43},\tau_{4321})$. We apply (8) with $\lambda^{m-3}=1$, $\lambda^{m-2}=42$, $|\lambda^{m-1}|\geq 2$, $\lambda^m=4321$. Then (11)–(12) vanish, (13) either vanishes or is known by previous cases, and (10) contributes $I_2(\tau_{\lambda^1},\dots,\tau_{\lambda^{m-4}},\tau_{\lambda^{m-1}},\tau_{43},\tau_{4321})+I_2(\tau_{\lambda^1},\dots,\tau_{\lambda^{m-4}},\tau_{\lambda^{m-1}},\tau_{421},\tau_{4321})$.

Case 6. Point and X_{42}: $I_2(\dots,\tau_{42},\tau_{4321})$. We apply (8) with $\lambda^{m-3}=1$, $\lambda^{m-2}=41$, $|\lambda^{m-1}|\geq 2$, $\lambda^m=4321$, and proceed as in Case 3.

Case 7. Point and X_{321}: $I_2(\dots,\tau_{321},\tau_{4321})$. We apply (8) with $\lambda^{m-3}=1$, $\lambda^{m-2}=32$, $|\lambda^{m-1}|\geq 2$, $\lambda^m=4321$, and proceed as in Case 5.

Case 8. Point and X_{41}: $I_2(\dots,\tau_{41},\tau_{4321})$. We apply (8) with $\lambda^{m-3}=1$, $\lambda^{m-2}=4$, $|\lambda^{m-1}|\geq 2$, $\lambda^m=4321$, and proceed as in Case 3.

Case 9. Point and X_{32}: $I_2(\dots,\tau_{32},\tau_{4321})$. We apply (8) with $\lambda^{m-3}=1$, $\lambda^{m-2}=31$, $|\lambda^{m-1}|\geq 2$, $\lambda^m=4321$, and proceed as in Case 5.

Case 10. Two line conditions: $I_2(\dots,\tau_{432},\tau_{432})$. We apply (8) with $\lambda^{m-3}=1$, $\lambda^{m-2}=431$, $|\lambda^{m-1}|\geq 2$, $\lambda^m=432$. Then (12) vanishes, (13) vanishes or is known by Case 2, (11) contributes $I_2(\tau_{\lambda^1},\dots,\tau_{\lambda^{m-4}},\tau_{\lambda^{m-1}},\tau_{431},\tau_{4321})$, and (10) contributes $I_2(\tau_{\lambda^1},\dots,\tau_{\lambda^{m-4}},\tau_{\lambda^{m-1}},\tau_{432},\tau_{432})$.

Case 11. Line and plane: $I_2(\dots,\tau_{431},\tau_{432})$. We apply (8) with $\lambda^{m-3}=1$, $\lambda^{m-2}=421$, $|\lambda^{m-1}|\geq 2$, $\lambda^m=432$, and proceed as in Case 10.

Case 12. Line and X_{421}: $I_2(\dots,\tau_{421},\tau_{432})$. We apply (8) with $\lambda^{m-3}=1$, $\lambda^{m-2}=321$, $|\lambda^{m-1}|\geq 2$, $\lambda^m=432$, and proceed as in Case 10.

Case 13. Line and X_{43}: $I_2(\dots,\tau_{43},\tau_{432})$. We apply (8) with $\lambda^{m-3}=1$, $\lambda^{m-2}=42$, $|\lambda^{m-1}|\geq 2$, $\lambda^m=432$. Then (12) vanishes, (13) vanishes or is known by previous cases, (11) contributes $I_2(\tau_{\lambda^1},\dots,\tau_{\lambda^{m-4}},\tau_{\lambda^{m-1}},\tau_{42},\tau_{4321})$, and (10) contributes $I_2(\tau_{\lambda^1},\dots,\tau_{\lambda^{m-4}},\tau_{\lambda^{m-1}},\tau_{43},\tau_{432})+I_2(\tau_{\lambda^1},\dots,\tau_{\lambda^{m-4}},\tau_{\lambda^{m-1}},\tau_{421},\tau_{432})$.

Case 14. Line and X_{42}: $I_2(\dots,\tau_{42},\tau_{432})$. We apply (8) with $\lambda^{m-3}=1$, $\lambda^{m-2}=41$, $|\lambda^{m-1}|\geq 2$, $\lambda^m=432$, and proceed as in Case 10.

Case 15. Line and X_{321}: $I_2(\dots,\tau_{321},\tau_{432})$. We apply (8) with $\lambda^{m-3}=1$, $\lambda^{m-2}=32$, $|\lambda^{m-1}|\geq 2$, $\lambda^m=432$, and proceed as in Case 13.

Case 16. Two plane conditions: $I_2(\dots,\tau_{431},\tau_{431})$. We apply (8) with $\lambda^{m-3}=1$, $\lambda^{m-2}=421$, $|\lambda^{m-1}|\geq 2$, $\lambda^m=431$. Then (12) vanishes or is known by Case 4, (11) is known by Case 12, (13) vanishes or is known by previous cases, and (10) contributes $I_2(\tau_{\lambda^1},\dots,\tau_{\lambda^{m-4}},\tau_{\lambda^{m-1}},\tau_{431},\tau_{431})$.

Case 17. Plane and X_{421}: $I_2(\dots,\tau_{421},\tau_{431})$. We apply (8) with $\lambda^{m-3}=1$, $\lambda^{m-2}=321$, $|\lambda^{m-1}|\geq 2$, $\lambda^m=431$. Then (11) is known by Case 15, (12) and (13) vanish or are known by previous cases, and (10) contributes $I_2(\tau_{\lambda^1},\dots,\tau_{\lambda^{m-4}},\tau_{\lambda^{m-1}},\tau_{421},\tau_{431})$.

We list a few of the conic numbers:

$$
\begin{aligned}
I_2(\tau_2,\tau_{421},\tau_{431},\tau_{4321})&=3, & I_2(\tau_3,\tau_{421},\tau_{431},\tau_{432})&=5,\\
I_2(\tau_{21},\tau_{421},\tau_{431},\tau_{432})&=4, & I_2(\tau_{421},\tau_{432},\tau_{4321})&=1,\\
I_2(\tau_{431},\tau_{431},\tau_{4321})&=1, & I_2(\tau_{431},\tau_{432},\tau_{432})&=2,
\end{aligned}
\tag{14}
$$

In total, Cases 1 through 17 determine 1459 conic numbers.

$I_3(\tau_2,\tau_2,\tau_2,\tau_2,\tau_{4321},\tau_{4321},\tau_{4321})=81,$	$I_3(\tau_2,\tau_2,\tau_2,\tau_3,\tau_{432},\tau_{4321},\tau_{4321})=216,$
$I_3(\tau_2,\tau_2,\tau_2,\tau_{21},\tau_{432},\tau_{4321},\tau_{4321})=135,$	$I_3(\tau_2,\tau_2,\tau_3,\tau_{4321},\tau_{4321},\tau_{4321})=18,$
$I_3(\tau_2,\tau_2,\tau_{21},\tau_{4321},\tau_{4321},\tau_{4321})=9,$	$I_3(\tau_2,\tau_2,\tau_4,\tau_{432},\tau_{4321},\tau_{4321})=24,$
$I_3(\tau_2,\tau_2,\tau_{31},\tau_{432},\tau_{4321},\tau_{4321})=42,$	$I_3(\tau_2,\tau_3,\tau_3,\tau_{432},\tau_{4321},\tau_{4321})=40,$
$I_3(\tau_2,\tau_3,\tau_{21},\tau_{432},\tau_{4321},\tau_{4321})=26,$	$I_3(\tau_2,\tau_{21},\tau_{21},\tau_{432},\tau_{4321},\tau_{4321})=16,$
$I_3(\tau_2,\tau_4,\tau_{4321},\tau_{4321},\tau_{4321})=3,$	$I_3(\tau_2,\tau_{31},\tau_{4321},\tau_{4321},\tau_{4321})=3,$
$I_3(\tau_2,\tau_{41},\tau_{432},\tau_{4321},\tau_{4321})=7,$	$I_3(\tau_2,\tau_{32},\tau_{432},\tau_{4321},\tau_{4321})=6,$
$I_3(\tau_3,\tau_3,\tau_3,\tau_{431},\tau_{4321},\tau_{4321})=52,$	$I_3(\tau_3,\tau_3,\tau_{21},\tau_{431},\tau_{4321},\tau_{4321})=36,$
$I_3(\tau_3,\tau_3,\tau_{4321},\tau_{4321},\tau_{4321})=4,$	$I_3(\tau_3,\tau_{21},\tau_{21},\tau_{431},\tau_{4321},\tau_{4321})=25,$
$I_3(\tau_3,\tau_{21},\tau_{4321},\tau_{4321},\tau_{4321})=2,$	$I_3(\tau_3,\tau_4,\tau_{432},\tau_{4321},\tau_{4321})=4,$
$I_3(\tau_3,\tau_{31},\tau_{432},\tau_{4321},\tau_{4321})=8,$	$I_3(\tau_3,\tau_{41},\tau_{431},\tau_{4321},\tau_{4321})=7,$
$I_3(\tau_3,\tau_{32},\tau_{431},\tau_{4321},\tau_{4321})=10,$	$I_3(\tau_{21},\tau_{21},\tau_{21},\tau_{431},\tau_{4321},\tau_{4321})=17,$
$I_3(\tau_{21},\tau_{21},\tau_{4321},\tau_{4321},\tau_{4321})=1,$	$I_3(\tau_{21},\tau_4,\tau_{432},\tau_{4321},\tau_{4321})=3,$
$I_3(\tau_{21},\tau_{31},\tau_{432},\tau_{4321},\tau_{4321})=5,$	$I_3(\tau_{21},\tau_{41},\tau_{431},\tau_{4321},\tau_{4321})=5,$
$I_3(\tau_{21},\tau_{32},\tau_{431},\tau_{4321},\tau_{4321})=7,$	$I_3(\tau_{41},\tau_{4321},\tau_{4321},\tau_{4321})=1,$
$I_3(\tau_{32},\tau_{4321},\tau_{4321},\tau_{4321})=0,$	$I_3(\tau_{42},\tau_{432},\tau_{4321},\tau_{4321})=2,$
$I_3(\tau_{321},\tau_{432},\tau_{4321},\tau_{4321})=0,$	$I_3(\tau_{43},\tau_{431},\tau_{4321},\tau_{4321})=1,$
$I_3(\tau_{421},\tau_{431},\tau_{4321},\tau_{4321})=2.$	

Table 2. Degree 3, Case 1 numbers

Example. *The number* $I_2(\tau_2,\tau_{421},\tau_{431},\tau_{4321})$ *falls under Case 3. We have* $m=5$, $\lambda^2=1$, $\lambda^3=421$, $\lambda^5=4321$, *and either* $\lambda^4=2$, *hence* $\lambda^1=421$ *with* (8) *giving*

$$\begin{aligned}I_2(\tau_2,&\tau_{421},\tau_{431},\tau_{4321})+I_1(\tau_1,\tau_4,\tau_{421},\tau_{421})I_1(\tau_2,\tau_{321},\tau_{4321})\\&+I_1(\tau_1,\tau_{31},\tau_{421},\tau_{421})I_1(\tau_2,\tau_{42},\tau_{4321})\\&=I_2(\tau_1,\tau_{421},\tau_{432},\tau_{4321})+I_1(\tau_1,\tau_{43},\tau_{4321})I_1(\tau_2,\tau_{21},\tau_{421},\tau_{421})\\&+I_1(\tau_1,\tau_{421},\tau_{4321})I_1(\tau_2,\tau_3,\tau_{421},\tau_{421})+I_1(\tau_1,\tau_1,\tau_{421},\tau_{4321})I_1(\tau_2,\tau_{421},\tau_{432}),\end{aligned}$$

or $\lambda^4=421$, *hence* $\lambda^1=2$ *and* (8) *giving*

$$\begin{aligned}I_2(\tau_2,&\tau_{421},\tau_{431},\tau_{4321})+I_1(\tau_1,\tau_2,\tau_{421},\tau_{432})I_1(\tau_1,\tau_{421},\tau_{4321})\\&=I_1(\tau_1,\tau_{43},\tau_{4321})I_1(\tau_2,\tau_{21},\tau_{421},\tau_{421})+I_1(\tau_1,\tau_{421},\tau_{4321})I_1(\tau_2,\tau_3,\tau_{421},\tau_{421})\\&+I_1(\tau_1,\tau_2,\tau_{42},\tau_{4321})I_1(\tau_{31},\tau_{421},\tau_{421})+I_1(\tau_1,\tau_2,\tau_{321},\tau_{4321})I_1(\tau_4,\tau_{421},\tau_{421}).\end{aligned}$$

Either way, we obtain $I_2(\tau_2,\tau_{421},\tau_{431},\tau_{4321})=3$. *One way requires the Case 2 number* $I_2(\tau_{421},\tau_{432},\tau_{4321})$. *The needed line numbers appear in* (7).

Higher degree numbers

The associativity relations also determine many higher-degree Gromov-Witten numbers. For instance, taking $d=3$ we may apply (8) with $\lambda^{m-3}=1$, $\lambda^{m-2}=432$, $|\lambda^{m-1}|\geq 2$, $\lambda^m=4321$ just as in Case 1 above, and obtain many Gromov-Witten numbers $I_3(\dots,\tau_{4321},\tau_{4321})$. However, since we obtained only *some* of the degree 2 Gromov-Witten numbers in the previous section, we need to check that the contributions with $d'=2$ or $d-d'=2$ involve only degree 2 Gromov-Witten

$$I_3(\tau_2,\tau_2,\tau_3,\tau_3,\tau_{432},\tau_{432},\tau_{4321}) = 548,\quad I_3(\tau_2,\tau_2,\tau_3,\tau_{21},\tau_{432},\tau_{432},\tau_{4321}) = 379,$$
$$I_3(\tau_2,\tau_2,\tau_{21},\tau_{21},\tau_{432},\tau_{432},\tau_{4321}) = 260,\quad I_3(\tau_2,\tau_2,\tau_{41},\tau_{432},\tau_{432},\tau_{4321}) = 80,$$
$$I_3(\tau_2,\tau_2,\tau_{32},\tau_{432},\tau_{432},\tau_{4321}) = 105,\quad I_3(\tau_2,\tau_3,\tau_3,\tau_3,\tau_{431},\tau_{432},\tau_{4321}) = 753,$$
$$I_3(\tau_2,\tau_3,\tau_3,\tau_{21},\tau_{431},\tau_{432},\tau_{4321}) = 531,\quad I_3(\tau_2,\tau_3,\tau_{21},\tau_{21},\tau_{431},\tau_{432},\tau_{4321}) = 377,$$
$$I_3(\tau_2,\tau_3,\tau_4,\tau_{432},\tau_{432},\tau_{4321}) = 47,\quad I_3(\tau_2,\tau_3,\tau_{31},\tau_{432},\tau_{432},\tau_{4321}) = 109,$$
$$I_3(\tau_2,\tau_3,\tau_{41},\tau_{431},\tau_{432},\tau_{4321}) = 96,\quad I_3(\tau_2,\tau_3,\tau_{32},\tau_{431},\tau_{432},\tau_{4321}) = 139,$$
$$I_3(\tau_2,\tau_{21},\tau_{21},\tau_{21},\tau_{431},\tau_{432},\tau_{4321}) = 270,\quad I_3(\tau_2,\tau_{21},\tau_4,\tau_{432},\tau_{432},\tau_{4321}) = 33,$$
$$I_3(\tau_2,\tau_{21},\tau_{31},\tau_{432},\tau_{432},\tau_{4321}) = 76,\quad I_3(\tau_2,\tau_{21},\tau_{41},\tau_{431},\tau_{432},\tau_{4321}) = 66,$$
$$I_3(\tau_2,\tau_{21},\tau_{32},\tau_{431},\tau_{432},\tau_{4321}) = 103,\quad I_3(\tau_2,\tau_{42},\tau_{432},\tau_{432},\tau_{4321}) = 22,$$
$$I_3(\tau_2,\tau_{321},\tau_{432},\tau_{432},\tau_{4321}) = 9,\quad I_3(\tau_2,\tau_{43},\tau_{431},\tau_{432},\tau_{4321}) = 19,$$
$$I_3(\tau_2,\tau_{421},\tau_{431},\tau_{432},\tau_{4321}) = 21,\quad I_3(\tau_3,\tau_3,\tau_3,\tau_{432},\tau_{432},\tau_{4321}) = 92,$$
$$I_3(\tau_3,\tau_3,\tau_{21},\tau_{432},\tau_{432},\tau_{4321}) = 64,\quad I_3(\tau_3,\tau_3,\tau_4,\tau_{431},\tau_{432},\tau_{4321}) = 59,$$
$$I_3(\tau_3,\tau_3,\tau_{31},\tau_{431},\tau_{432},\tau_{4321}) = 142,\quad I_3(\tau_3,\tau_{21},\tau_{21},\tau_{432},\tau_{432},\tau_{4321}) = 45,$$
$$I_3(\tau_3,\tau_{21},\tau_4,\tau_{431},\tau_{432},\tau_{4321}) = 41,\quad I_3(\tau_3,\tau_{21},\tau_{31},\tau_{431},\tau_{432},\tau_{4321}) = 101,$$
$$I_3(\tau_3,\tau_{41},\tau_{432},\tau_{432},\tau_{4321}) = 13,\quad I_3(\tau_3,\tau_{32},\tau_{432},\tau_{432},\tau_{4321}) = 18,$$
$$I_3(\tau_3,\tau_{42},\tau_{431},\tau_{432},\tau_{4321}) = 25,\quad I_3(\tau_3,\tau_{321},\tau_{431},\tau_{432},\tau_{4321}) = 11,$$
$$I_3(\tau_{21},\tau_{21},\tau_{21},\tau_{432},\tau_{432},\tau_{4321}) = 31,\quad I_3(\tau_{21},\tau_{21},\tau_4,\tau_{431},\tau_{432},\tau_{4321}) = 28,$$
$$I_3(\tau_{21},\tau_{21},\tau_{31},\tau_{431},\tau_{432},\tau_{4321}) = 73,\quad I_3(\tau_{21},\tau_{41},\tau_{432},\tau_{432},\tau_{4321}) = 9,$$
$$I_3(\tau_{21},\tau_{32},\tau_{432},\tau_{432},\tau_{4321}) = 13,\quad I_3(\tau_{21},\tau_{42},\tau_{431},\tau_{432},\tau_{4321}) = 17,$$
$$I_3(\tau_{21},\tau_{321},\tau_{431},\tau_{432},\tau_{4321}) = 10,\quad I_3(\tau_4,\tau_4,\tau_{432},\tau_{432},\tau_{4321}) = 4,$$
$$I_3(\tau_4,\tau_{31},\tau_{432},\tau_{432},\tau_{4321}) = 9,\quad I_3(\tau_4,\tau_{41},\tau_{431},\tau_{432},\tau_{4321}) = 8,$$
$$I_3(\tau_4,\tau_{32},\tau_{431},\tau_{432},\tau_{4321}) = 9,\quad I_3(\tau_{31},\tau_{31},\tau_{432},\tau_{432},\tau_{4321}) = 22,$$
$$I_3(\tau_{31},\tau_{41},\tau_{431},\tau_{432},\tau_{4321}) = 17,\quad I_3(\tau_{31},\tau_{32},\tau_{431},\tau_{432},\tau_{4321}) = 27,$$
$$I_3(\tau_{43},\tau_{432},\tau_{432},\tau_{4321}) = 3,\quad I_3(\tau_{421},\tau_{432},\tau_{432},\tau_{4321}) = 3,$$
$$I_3(\tau_{431},\tau_{431},\tau_{432},\tau_{4321}) = 5.$$

Table 3. Degree 3, Case 2 numbers

$$I_3(\tau_3,\tau_3,\tau_3,\tau_3,\tau_{431},\tau_{431},\tau_{4321}) = 1062,\quad I_3(\tau_3,\tau_3,\tau_3,\tau_{21},\tau_{431},\tau_{431},\tau_{4321}) = 750,$$
$$I_3(\tau_3,\tau_3,\tau_{21},\tau_{21},\tau_{431},\tau_{431},\tau_{4321}) = 534,\quad I_3(\tau_3,\tau_3,\tau_{41},\tau_{431},\tau_{431},\tau_{4321}) = 120,$$
$$I_3(\tau_3,\tau_3,\tau_{32},\tau_{431},\tau_{431},\tau_{4321}) = 174,\quad I_3(\tau_3,\tau_{21},\tau_{21},\tau_{21},\tau_{431},\tau_{431},\tau_{4321}) = 385,$$
$$I_3(\tau_3,\tau_{21},\tau_{41},\tau_{431},\tau_{431},\tau_{4321}) = 83,\quad I_3(\tau_3,\tau_{21},\tau_{32},\tau_{431},\tau_{431},\tau_{4321}) = 128,$$
$$I_3(\tau_3,\tau_{43},\tau_{431},\tau_{431},\tau_{4321}) = 20,\quad I_3(\tau_3,\tau_{421},\tau_{431},\tau_{431},\tau_{4321}) = 21,$$
$$I_3(\tau_{21},\tau_{21},\tau_{21},\tau_{21},\tau_{431},\tau_{431},\tau_{4321}) = 282,\quad I_3(\tau_{21},\tau_{21},\tau_{41},\tau_{431},\tau_{431},\tau_{4321}) = 58,$$
$$I_3(\tau_{21},\tau_{21},\tau_{32},\tau_{431},\tau_{431},\tau_{4321}) = 95,\quad I_3(\tau_{21},\tau_{43},\tau_{431},\tau_{431},\tau_{4321}) = 13,$$
$$I_3(\tau_{21},\tau_{421},\tau_{431},\tau_{431},\tau_{4321}) = 16,\quad I_3(\tau_{41},\tau_{41},\tau_{431},\tau_{431},\tau_{4321}) = 12,$$
$$I_3(\tau_{41},\tau_{32},\tau_{431},\tau_{431},\tau_{4321}) = 17,\quad I_3(\tau_{32},\tau_{32},\tau_{431},\tau_{431},\tau_{4321}) = 28.$$

Table 4. Degree 3, Case 3 numbers

numbers that have been determined. This is checked on a case-by-case basis for each of the 35 numbers listed in Table 2 and each corresponding application of (8).

$$
\begin{aligned}
I_3(\tau_2,\tau_3,\tau_3,\tau_3,\tau_{432},\tau_{432},\tau_{432}) &= 1416, & I_3(\tau_2,\tau_3,\tau_3,\tau_{21},\tau_{432},\tau_{432},\tau_{432}) &= 996,\\
I_3(\tau_2,\tau_3,\tau_{21},\tau_{21},\tau_{432},\tau_{432},\tau_{432}) &= 708, & I_3(\tau_2,\tau_3,\tau_{41},\tau_{432},\tau_{432},\tau_{432}) &= 189,\\
I_3(\tau_2,\tau_3,\tau_{32},\tau_{432},\tau_{432},\tau_{432}) &= 270, & I_3(\tau_2,\tau_{21},\tau_{21},\tau_{21},\tau_{432},\tau_{432},\tau_{432}) &= 510,\\
I_3(\tau_2,\tau_{21},\tau_{41},\tau_{432},\tau_{432},\tau_{432}) &= 129, & I_3(\tau_2,\tau_{21},\tau_{32},\tau_{432},\tau_{432},\tau_{432}) &= 201,\\
I_3(\tau_2,\tau_{43},\tau_{432},\tau_{432},\tau_{432}) &= 42, & I_3(\tau_2,\tau_{421},\tau_{432},\tau_{432},\tau_{432}) &= 42,\\
I_3(\tau_3,\tau_3,\tau_3,\tau_3,\tau_{431},\tau_{432},\tau_{432}) &= 1940, & I_3(\tau_3,\tau_3,\tau_3,\tau_{21},\tau_{431},\tau_{432},\tau_{432}) &= 1362,\\
I_3(\tau_3,\tau_3,\tau_{21},\tau_{21},\tau_{431},\tau_{432},\tau_{432}) &= 966, & I_3(\tau_3,\tau_3,\tau_4,\tau_{432},\tau_{432},\tau_{432}) &= 112,\\
I_3(\tau_3,\tau_3,\tau_{31},\tau_{432},\tau_{432},\tau_{432}) &= 268, & I_3(\tau_3,\tau_3,\tau_{41},\tau_{431},\tau_{432},\tau_{432}) &= 228,\\
I_3(\tau_3,\tau_3,\tau_{32},\tau_{431},\tau_{432},\tau_{432}) &= 320, & I_3(\tau_3,\tau_{21},\tau_{21},\tau_{21},\tau_{431},\tau_{432},\tau_{432}) &= 696,\\
I_3(\tau_3,\tau_{21},\tau_4,\tau_{432},\tau_{432},\tau_{432}) &= 77, & I_3(\tau_3,\tau_{21},\tau_{31},\tau_{432},\tau_{432},\tau_{432}) &= 191,\\
I_3(\tau_3,\tau_{21},\tau_{41},\tau_{431},\tau_{432},\tau_{432}) &= 156, & I_3(\tau_3,\tau_{21},\tau_{32},\tau_{431},\tau_{432},\tau_{432}) &= 236,\\
I_3(\tau_3,\tau_{42},\tau_{432},\tau_{432},\tau_{432}) &= 50, & I_3(\tau_3,\tau_{321},\tau_{432},\tau_{432},\tau_{432}) &= 22,\\
I_3(\tau_3,\tau_{43},\tau_{431},\tau_{432},\tau_{432}) &= 42, & I_3(\tau_3,\tau_{421},\tau_{431},\tau_{432},\tau_{432}) &= 39,\\
I_3(\tau_{21},\tau_{21},\tau_{21},\tau_{21},\tau_{431},\tau_{432},\tau_{432}) &= 512, & I_3(\tau_{21},\tau_{21},\tau_4,\tau_{432},\tau_{432},\tau_{432}) &= 52,\\
I_3(\tau_{21},\tau_{21},\tau_{31},\tau_{432},\tau_{432},\tau_{432}) &= 139, & I_3(\tau_{21},\tau_{21},\tau_{41},\tau_{431},\tau_{432},\tau_{432}) &= 108,\\
I_3(\tau_{21},\tau_{21},\tau_{32},\tau_{431},\tau_{432},\tau_{432}) &= 176, & I_3(\tau_{21},\tau_{42},\tau_{432},\tau_{432},\tau_{432}) &= 34,\\
I_3(\tau_{21},\tau_{321},\tau_{432},\tau_{432},\tau_{432}) &= 20, & I_3(\tau_{21},\tau_{43},\tau_{431},\tau_{432},\tau_{432}) &= 27,\\
I_3(\tau_{21},\tau_{421},\tau_{431},\tau_{432},\tau_{432}) &= 30, & I_3(\tau_4,\tau_{41},\tau_{432},\tau_{432},\tau_{432}) &= 16,\\
I_3(\tau_4,\tau_{32},\tau_{432},\tau_{432},\tau_{432}) &= 18, & I_3(\tau_{31},\tau_{41},\tau_{432},\tau_{432},\tau_{432}) &= 34,\\
I_3(\tau_{31},\tau_{32},\tau_{432},\tau_{432},\tau_{432}) &= 54, & I_3(\tau_{41},\tau_{41},\tau_{431},\tau_{432},\tau_{432}) &= 24,\\
I_3(\tau_{41},\tau_{32},\tau_{431},\tau_{432},\tau_{432}) &= 33, & I_3(\tau_{32},\tau_{32},\tau_{431},\tau_{432},\tau_{432}) &= 54,\\
I_3(\tau_{431},\tau_{432},\tau_{432},\tau_{432}) &= 11.
\end{aligned}
$$

TABLE 5. Degree 3, Case 10 numbers

$$
\begin{aligned}
I_4(\tau_2,\tau_2,\tau_2,\tau_{4321},\tau_{4321},\tau_{4321},\tau_{4321}) &= 64, & I_4(\tau_2,\tau_2,\tau_3,\tau_{432},\tau_{4321},\tau_{4321},\tau_{4321}) &= 208,\\
I_4(\tau_2,\tau_2,\tau_{21},\tau_{432},\tau_{4321},\tau_{4321},\tau_{4321}) &= 160, & I_4(\tau_2,\tau_3,\tau_3,\tau_{432},\tau_{432},\tau_{4321},\tau_{4321}) &= 576,\\
I_4(\tau_2,\tau_3,\tau_{21},\tau_{432},\tau_{432},\tau_{4321},\tau_{4321}) &= 420, & I_4(\tau_2,\tau_3,\tau_{4321},\tau_{4321},\tau_{4321},\tau_{4321}) &= 8,\\
I_4(\tau_2,\tau_{21},\tau_{21},\tau_{432},\tau_{432},\tau_{4321},\tau_{4321}) &= 304, & I_4(\tau_2,\tau_{21},\tau_{4321},\tau_{4321},\tau_{4321},\tau_{4321}) &= 8,\\
I_4(\tau_2,\tau_4,\tau_{432},\tau_{4321},\tau_{4321},\tau_{4321}) &= 12, & I_4(\tau_2,\tau_{31},\tau_{432},\tau_{4321},\tau_{4321},\tau_{4321}) &= 38,\\
I_4(\tau_2,\tau_{41},\tau_{432},\tau_{432},\tau_{4321},\tau_{4321}) &= 62, & I_4(\tau_2,\tau_{32},\tau_{432},\tau_{432},\tau_{4321},\tau_{4321}) &= 100,\\
I_4(\tau_3,\tau_3,\tau_{432},\tau_{4321},\tau_{4321},\tau_{4321}) &= 28, & I_4(\tau_3,\tau_{21},\tau_{432},\tau_{4321},\tau_{4321},\tau_{4321}) &= 22,\\
I_4(\tau_3,\tau_4,\tau_{432},\tau_{432},\tau_{4321},\tau_{4321}) &= 36, & I_4(\tau_3,\tau_{31},\tau_{432},\tau_{432},\tau_{4321},\tau_{4321}) &= 94,\\
I_4(\tau_{21},\tau_{21},\tau_{432},\tau_{4321},\tau_{4321},\tau_{4321}) &= 16, & I_4(\tau_{21},\tau_4,\tau_{432},\tau_{432},\tau_{4321},\tau_{4321}) &= 26,\\
I_4(\tau_{21},\tau_{31},\tau_{432},\tau_{432},\tau_{4321},\tau_{4321}) &= 68, & I_4(\tau_4,\tau_{4321},\tau_{4321},\tau_{4321},\tau_{4321}) &= 0,\\
I_4(\tau_{31},\tau_{4321},\tau_{4321},\tau_{4321},\tau_{4321}) &= 2, & I_4(\tau_{41},\tau_{432},\tau_{4321},\tau_{4321},\tau_{4321}) &= 3,\\
I_4(\tau_{32},\tau_{432},\tau_{4321},\tau_{4321},\tau_{4321}) &= 6, & I_4(\tau_{42},\tau_{432},\tau_{432},\tau_{4321},\tau_{4321}) &= 14,\\
I_4(\tau_{321},\tau_{432},\tau_{432},\tau_{4321},\tau_{4321}) &= 8.
\end{aligned}
$$

TABLE 6. Degree 4, Case 1 numbers

Example. *To determine* $I_3(\tau_{421},\tau_{431},\tau_{4321},\tau_{4321})$ *we read off from* (8) *with* $d=3$, $m=5$, *and* $(\lambda^1,\dots,\lambda^5)=(431,1,432,421,4321)$, *the identity (cf.* (7), (14)*):*

$$
\begin{aligned}
&I_3(\tau_{421},\tau_{431},\tau_{4321},\tau_{4321})\\
&\quad= I_1(\tau_1,\tau_{43},\tau_{4321})I_2(\tau_{21},\tau_{421},\tau_{431},\tau_{432}) + I_1(\tau_1,\tau_{421},\tau_{4321})I_2(\tau_3,\tau_{421},\tau_{431},\tau_{432})\\
&\quad+ I_2(\tau_1,\tau_{431},\tau_{431},\tau_{4321})I_1(\tau_2,\tau_{421},\tau_{432}) - I_1(\tau_1,\tau_{431},\tau_{432})I_2(\tau_2,\tau_{421},\tau_{431},\tau_{4321})\\
&\quad- I_1(\tau_1,\tau_1,\tau_{431},\tau_{432})I_2(\tau_{421},\tau_{432},\tau_{4321}) - I_2(\tau_1,\tau_{431},\tau_{432},\tau_{432})I_1(\tau_1,\tau_{421},\tau_{4321})\\
&\quad= 1\cdot 4 + 0\cdot 5 + 2\cdot 1 - 1\cdot 3 - 1\cdot 1 - 4\cdot 0 = 2.
\end{aligned}
$$

$$I_4(\tau_3,\tau_3,\tau_3,\tau_{432},\tau_{432},\tau_{432},\tau_{4321}) = 1488, \qquad I_4(\tau_3,\tau_3,\tau_{21},\tau_{432},\tau_{432},\tau_{432},\tau_{4321}) = 1062,$$
$$I_4(\tau_3,\tau_{21},\tau_{21},\tau_{432},\tau_{432},\tau_{432},\tau_{4321}) = 764, \qquad I_4(\tau_3,\tau_{41},\tau_{432},\tau_{432},\tau_{432},\tau_{4321}) = 154,$$
$$I_4(\tau_3,\tau_{32},\tau_{432},\tau_{432},\tau_{432},\tau_{4321}) = 232, \qquad I_4(\tau_{21},\tau_{21},\tau_{21},\tau_{432},\tau_{432},\tau_{432},\tau_{4321}) = 552,$$
$$I_4(\tau_{21},\tau_{41},\tau_{432},\tau_{432},\tau_{432},\tau_{4321}) = 108, \qquad I_4(\tau_{21},\tau_{32},\tau_{432},\tau_{432},\tau_{432},\tau_{4321}) = 170,$$
$$I_4(\tau_{43},\tau_{432},\tau_{432},\tau_{432},\tau_{4321}) = 26, \qquad I_4(\tau_{421},\tau_{432},\tau_{432},\tau_{432},\tau_{4321}) = 29.$$

Table 7. Degree 4, Case 2 numbers

$$I_5(\tau_2,\tau_2,\tau_{4321},\dots,\tau_{4321}) = 125, \qquad I_5(\tau_2,\tau_3,\tau_{432},\tau_{4321},\dots,\tau_{4321}) = 250,$$
$$I_5(\tau_2,\tau_{21},\tau_{432},\tau_{4321},\dots,\tau_{4321}) = 175, \qquad I_5(\tau_3,\tau_3,\tau_{432},\tau_{432},\tau_{4321},\dots,\tau_{4321}) = 566,$$
$$I_5(\tau_3,\tau_{21},\tau_{432},\tau_{432},\tau_{4321},\dots,\tau_{4321}) = 403, \qquad I_5(\tau_3,\tau_{4321},\dots,\tau_{4321}) = 15,$$
$$I_5(\tau_{21},\tau_{21},\tau_{432},\tau_{432},\tau_{4321},\dots,\tau_{4321}) = 288, \qquad I_5(\tau_{21},\tau_{4321},\dots,\tau_{4321}) = 10,$$
$$I_5(\tau_4,\tau_{432},\tau_{4321},\dots,\tau_{4321}) = 14, \qquad I_5(\tau_{31},\tau_{432},\tau_{4321},\dots,\tau_{4321}) = 33,$$
$$I_5(\tau_{41},\tau_{432},\tau_{432},\tau_{4321},\dots,\tau_{4321}) = 50, \qquad I_5(\tau_{32},\tau_{432},\tau_{432},\tau_{4321},\dots,\tau_{4321}) = 75,$$
$$I_6(\tau_2,\tau_{4321},\dots,\tau_{4321}) = 60, \qquad I_6(\tau_3,\tau_{432},\tau_{4321},\dots,\tau_{4321}) = 180,$$
$$I_6(\tau_{21},\tau_{432},\tau_{4321},\dots,\tau_{4321}) = 130, \qquad I_7(\tau_{4321},\dots,\tau_{4321}) = 71.$$

Table 8. Degree 5, 6, and 7 numbers (all Case 1)

Alternatively $(\lambda^1,\dots,\lambda^5) = (421, 1, 432, 431, 4321)$ *yields* $I_3(\tau_{421},\tau_{431},\tau_{4321},\tau_{4321}) = 1\cdot 4 + 0\cdot 5 + 0\cdot 2 + 2\cdot 1 - 1\cdot 3 - 1\cdot 1 = 2.$

Reasoning as in Case 2 we obtain the Gromov-Witten numbers $I_3(\dots,\tau_{432},\tau_{4321})$ listed in Table 3. Again it must be checked that each application of (8) requires only known conic numbers.

Similarly we reason as in Case 3 above to obtain the $I_3(\dots,\tau_{431},\tau_{4321})$ listed in Table 4. We conclude our determination of $d = 3$ numbers with the $I_3(\dots,\tau_{432},\tau_{432})$ listed in Table 5, for which the reasoning is as in Case 10.

An application of (8) with $d = 4$ requires numbers of degrees 1, 2, and 3. It must be verified on a case-by-case basis that the required conic and cubic numbers are among those already determined. Tables 6 and 7 list the numbers $I_4(\dots,\tau_{4321},\tau_{4321})$, respectively $I_4(\dots,\tau_{432},\tau_{4321})$, which are treated by reasoning as in Case 1 and Case 2, respectively. For $d = 5$, 6, and 7 we require only numbers with at least two point conditions, hence we use the reasoning of Case 1. Again it must be verified on a case-by-case basis that the required numbers of every smaller degree are among those already determined. The numbers are displayed in Table 8. The final number displayed is the desired

$$I_7(\tau_{4321},\tau_{4321},\tau_{4321},\tau_{4321},\tau_{4321},\tau_{4321},\tau_{4321}) = 71,$$

with the following enumerative interpretation.

Proposition. *There are* 71 *rational curves of degree* 7 *through* 7 *general points on* $\mathrm{OG}(5,10)$.

Remark. Semi-simplicity allows us to reconstruct the full quantum cohomology even without assuming that the ordinary cohomology is generated by H^2, see [BM04]

and [Mas11]. (The case where the cohomology is generated by H^2 is addressed in [KM94].) This property was verified for orthogonal Grassmannians in [CMP10].

References

[BGG73] I. N. Bernšteĭn, I. M. Gel′fand, and S. I. Gel′fand. Schubert cells, and the cohomology of the spaces G/P. *Uspehi Mat. Nauk*, 28(3(171)):3–26, 1973. English transl.: Russian Math. Surveys 28 (1973), no. 3, 1–26.

[BH95] S. Billey and M. Haiman. Schubert polynomials for the classical groups. *J. Amer. Math. Soc.*, 8(2):443–482, 1995.

[BM04] A. Bayer and Y. I. Manin. (Semi)simple exercises in quantum cohomology. In *The Fano Conference*, pages 143–173. Univ. Torino, Turin, 2004.

[CMP10] P. E. Chaput, L. Manivel, and N. Perrin. Quantum cohomology of minuscule homogeneous spaces III. Semi-simplicity and consequences. *Canad. J. Math.*, 62(6):1246–1263, 2010.

[Dem74] M. Demazure. Désingularisation des variétés de Schubert généralisées. *Ann. Sci. École Norm. Sup. (4)*, 7:53–88, 1974. Collection of articles dedicated to Henri Cartan on the occasion of his 70th birthday, I.

[FP97] W. Fulton and R. Pandharipande. Notes on stable maps and quantum cohomology. In *Algebraic geometry—Santa Cruz 1995*, volume 62 of *Proc. Sympos. Pure Math.*, pages 45–96. Amer. Math. Soc., Providence, RI, 1997.

[HB86] H. Hiller and B. Boe. Pieri formula for $\mathrm{SO}_{2n+1}/\mathrm{U}_n$ and $\mathrm{Sp}_n/\mathrm{U}_n$. *Adv. in Math.*, 62(1):49–67, 1986.

[KM94] M. Kontsevich and Yu. Manin. Gromov-Witten classes, quantum cohomology, and enumerative geometry. *Comm. Math. Phys.*, 164(3):525–562, 1994.

[KT04] A. Kresch and H. Tamvakis. Quantum cohomology of orthogonal Grassmannians. *Compos. Math.*, 140(2):482–500, 2004.

[LM03] J. M. Landsberg and L. Manivel. On the projective geometry of rational homogeneous varieties. *Comment. Math. Helv.*, 78(1):65–100, 2003.

[LM11] A. López Martín. Gromov-Witten invariants and rational curves on Grassmannians. *J. Geom. Phys.*, 61(1):213–216, 2011.

[Mas11] T. Maszczyk. Computing genus zero Gromov-Witten invariants of Fano varieties. *J. Geom. Phys.*, 61(6):1079–1092, 2011.

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